Nash equilibrium in games with continuous action spaces

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Games with Continuous Actions Spaces

- So far, we considered that players select one among a discrete list of available actions, e.g., $s_i \in \{Enter, NotEnter\}$, $s_i \in \{x, y, z\}$.
- But in some economic settings, agents can select among an infinite list of actions.
  - **Examples:** an output level $q_i \in \mathbb{R}_+$ (as in the Cournot game of output competition),
  - A price level $p_i \in \mathbb{R}_+$ (as in the Bertrand game of price competition),
  - Contribution $c_i \in \mathbb{R}_+$ to a charity in a public good game,
  - Exploitation level $x_i \in \mathbb{R}_+$ of a common pool resource, etc.

Cournot Game of Output Competition

- We first assume that $N = 2$ firms compete selling a homogenous product (no product differentiation).
  - Later on (maybe in a homework) you will analyze the case where firms sell differentiated products (easy! don't worry).
- Firm i's total cost function is $TC_i(q_i) = c_i q_i$.
  - Note that this allows for firms to be symmetric in costs, $c_i = c_j$, or asymmetric, $c_i > c_j$.
- Inverse demand function is linear $p(Q) = a - bQ$, where $Q = q_1 + q_2$ denotes the aggregate output, $a > c$ and $b > 0$.

Cournot Game of Output Competition

- Since $p(Q) = a - bQ$, where $Q = q_1 + q_2$, the profit maximization problem for firm 1 is therefore

$$\begin{aligned}
\max_{q_1} \quad \pi_1(q_1, q_2) &= [a - b(q_1 + q_2)]q_1 - c_1 q_1 \\
&= aq_1 - b(q_1 + q_2)q_1 - c_1 q_1 \\
&= aq_1 - bq_1^2 - bq_1 q_2 - c_1 q_1
\end{aligned}$$

Cournot Game of Output Competition

- Taking first-order conditions with respect to q_1,

$$a - 2bq_1 - bq_2 - c_1 = 0$$

and solving for q_1, we obtain

$$q_1 = \frac{a - c_1}{2b} - \frac{1}{2}q_2$$

Cournot Game of Output Competition

- Using $q_1 = \frac{a-c_1}{2b} - \frac{1}{2}q_2$, note that:
  - q_1 is positive when $q_2 = 0$, i.e., $q_1 = \frac{a-c_1}{2b}$, but...
  - q_1 decreases in q_2, becoming zero when q_2 is sufficiently large. In particular, $q_1 = 0$, when

$$0 = \frac{a - c_1}{2b} - \frac{1}{2}q_2 \implies \frac{a - c_1}{b} = q_2$$

Cournot Game of Output Competition

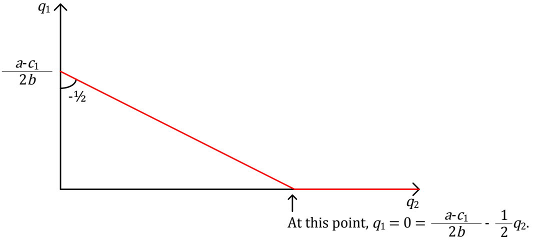
- We can hence, report firm 1's profit maximizing output as follows

$$q_1(q_2) = \begin{cases} \frac{a-c_1}{2b} - \frac{1}{2}q_2 \text{ if } q_2 \leq \frac{a-c_1}{b} \\ 0 \qquad \text{if } q_2 > \frac{a-c_1}{b} \end{cases}$$

- This is firm 1's **best response function**: it tells firm 1 how many units to produce in order to maximize profits as a function of firm 2's output, q_2 [See figure].

Cournot Game of Output Competition

- Drawing a single BRF: $q_1(q_2) = \begin{cases} \frac{a-c_1}{2b} - \frac{1}{2}q_2 & \text{if } q_2 \leq \frac{a-c_1}{b} \\ 0 & \text{if } q_2 > \frac{a-c_1}{b} \end{cases}$



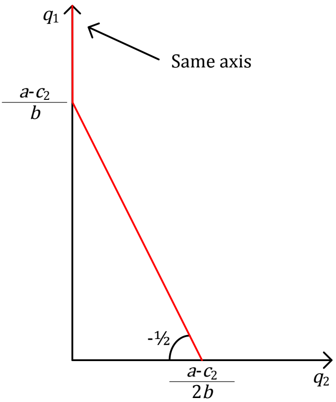
- In order to find the horizontal intercept, where $q_1 = 0$, we solve for q_2, as follows

$$0 = \frac{a-c_1}{2b} - \frac{1}{2}q_2 \implies \frac{a-c_1}{b} = q_2$$

- Hence, the horizontal intercept of BRF_1 is $q_2 = \frac{a-c_1}{b}$

Cournot Game of Output Competition

- Similarly for BRF_2: $q_2(q_1) = \begin{cases} \frac{a-c_2}{2b} - \frac{1}{2}q_2 & \text{if } q_1 \leq \frac{a-c_2}{b} \\ 0 & \text{if } q_1 > \frac{a-c_2}{b} \end{cases}$
- Note that we depict BRF_2 using the same axis as for BRF_1 in order to superimpose both BRFs later on.

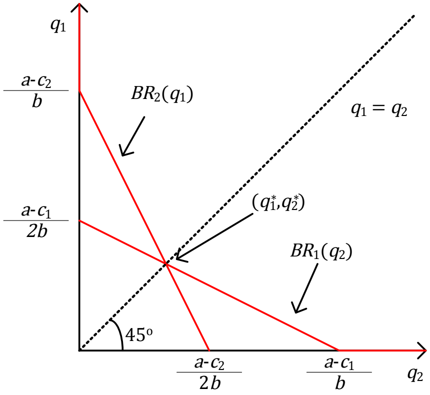


Cournot Game of Output Competition

- Putting both firms' *BRF* together... we obtain two figures:
  - one for the case in which firms are symmetric in marginal costs, $c_1 = c_2$, and
  - another figure for the case in which firms are asymmetric, $c_2 > c_1$.

Cournot Game of Output Competition

- If $c_1 = c_2$, (firms are symmetric in costs),



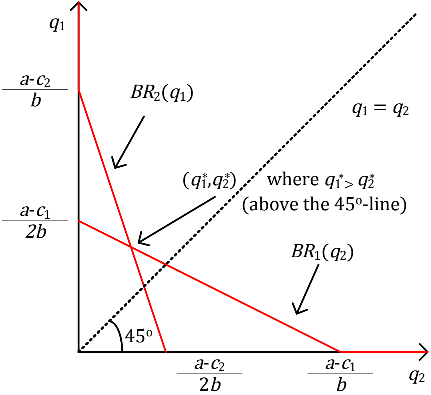
Cournot Game of Output Competition

- Since $c_1 = c_2$, then

$$\begin{aligned} \frac{a-c_1}{2b} &= \frac{a-c_2}{2b} \text{ (vertical intercepts)} \\ \frac{a-c_1}{b} &= \frac{a-c_2}{b} \text{ (horizontal intercepts)} \end{aligned}$$

Cournot Game of Output Competition

- If $c_2 > c_1$ (firm 1 is more competitive),



Cournot Game of Output Competition

- Since $c_2 > c_1$,

$$\begin{aligned} \frac{a-c_1}{2b} &> \frac{a-c_2}{2b} \text{ (vertical intercepts)} \\ \frac{a-c_1}{b} &> \frac{a-c_2}{b} \text{ (horizontal intercepts)} \end{aligned}$$

Cournot Game of Output Competition

- How can we find the NE of this game?
  - We know that each firm must be using its BRF in equilibrium.
  - We must then find the point where BRF_1 and BRF_2 cross each other.
  - Assuming an interior solution,

$$BRF_1 \longrightarrow q_1 = \frac{a-c_1}{2b} - \frac{1}{2}q_2 = \frac{a-c_1}{2b} - \frac{1}{2}\left(\underbrace{\frac{a-c_2}{2b} - \frac{1}{2}q_1}_{BRF_2}\right)$$

and solving for q_1,

$$q_1 = \frac{a-2c_1+c_2}{3b}$$

Similarly for q_2,

$$q_2 = \frac{a-2c_2+c_1}{3b}$$

Cournot Game of Output Competition

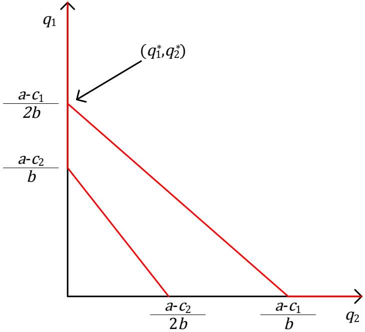
- What about Corner Solutions?
  - Using the figures, we can easily determine a condition for firm 2's equilibrium output, q_2^*, to be zero...
  - In particular, the horizontal intercept of firm 2's *BRF* lies below the vertical intercept of firm 1's *BRF*.
    - That is, if

$$\frac{a - c_2}{b} < \frac{a - c_1}{2b} \iff \frac{a + c_1}{2} < c_2$$

- As depicted in the next figure

Cournot Game of Output Competition

- Corner Solution with only firm 1 producing



- Note that (q_1^*, q_2^*) is the only crossing point between BRF_1 and BRF_2, implying $q_1^* > 0$, but $q_2^* = 0$.

Cournot Game of Output Competition

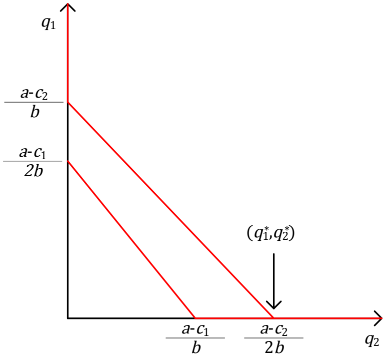
- This corner solution happens when

$$\frac{a - c_2}{b} < \frac{a - c_1}{2b} \iff \frac{a + c_1}{2} < c_2$$

- **Intuition:** Firm 1 is super-competitive (High c_2).

Cournot Game of Output Competition

- Another Corner Solution with only firm 2 producing:



- Note that (q_1^*, q_2^*) is the only crossing point between BRF_1 and BRF_2, implying $q_2^* > 0$, but $q_1^* = 0$.

Cournot Game of Output Competition

- This corner solution happens when

$$\frac{a - c_2}{b} > \frac{a - c_1}{2b} \iff \frac{a + c_1}{2} > c_2$$

- **Intuition:** Firm 2 is super-competitive (Low c_2).

Cournot Game of Output Competition

- Hence, aggregate output (assuming interior solutions) is

$$Q = q_1 + q_2 = \frac{a - 2c_1 + c_2}{3b} + \frac{a - 2c_2 + c_1}{3b} = \frac{2a - c_1 - c_2}{3b}$$

and the equilibrium price is

$$p = a - bQ = a - b\left(\underbrace{\frac{2a - c_1 - c_2}{3b}}_{Q}\right) = \frac{a + c_1 + c_2}{3}.$$

- Assuming symmetry ($c_1 = c_2 = c$), profits are

$$\pi_i = (p - c)q_i = \left(\frac{a + 2c}{3} - c\right)\frac{a - c}{3b} = \frac{(a - c)^2}{9b}$$

- **Practice**: find profits *without symmetry*. If we assume that $c_2 > c_1$, which firm experiences the highest profit?

Cournot Game of Output Competition

- This is very similar to the prisoner's dilemma!
- Indeed, if firms coordinate their production to lower production levels, they would maximize their joint profits.
  - Let us show how (for simplicity we assume symmetry in costs).
- First, note that firms would maximize their joint profits by choosing q_1 and q_2 such that

$$\begin{aligned}
\max \quad \pi_1 + \pi_2 &= [(a - b(q_1 + q_2))q_1 - cq_1] \\
&\quad + [(a - b(q_1 + q_2))q_2 - cq_2] \\
&= (a - bQ)Q - cQ \\
&= aQ - bQ^2 - cQ
\end{aligned}$$

Cournot Game of Output Competition

- Taking first-order conditions with respect to Q, we obtain

$$a - 2bQ - c = 0$$

  and solving for Q,we obtain the aggregate output level for the cartel

$$Q = \frac{a - c}{2b}$$

- Since firms are symmetric in costs, each produces half of this aggregate output level,

$$q_i = \frac{1}{2}\frac{a - c}{2b}$$

Cournot Game of Output Competition

Hence, equilibrium price is

$$p = a - bQ = a - b\left(\frac{a-c}{2b}\right) = \frac{a+c}{2}$$

and profits for every firm i are

$$\pi_i = p \cdot q_i - cq_i = \frac{a+c}{2}\left(\frac{a-c}{2b}\right) - c\left(\frac{a-c}{4b}\right) = \frac{(a-c)^2}{8b}$$

which is higher than the individual profit for every firm under Cournot competition, $\frac{(a-c)^2}{9b}$.

Cournot Game of Output Competition

- What if my firm deviates to Cournot output?

$$\pi_i = pq_i - cq_i = \left[a - b\left(\underbrace{\frac{a-c}{3b}}_{q_i^{Cournot}} + \underbrace{\frac{a-c}{4b}}_{q_j^{Cartel}}\right)\right] \cdot \frac{a-c}{3b}$$

$$-c\left(\frac{a-c}{3b}\right)$$

$$= \frac{5(a-c)^2}{36b}$$

(and Firm j makes a profit of $\frac{5(a-c)^2}{48b}$).

Cournot Game of Output Competition

- Putting everything together:

Firm 2

Firm 1	Participate in Cartel	Compete in Quantities
Participate in Cartel	$\frac{(a-c)^2}{8b}, \frac{(a-c)^2}{8b}$	$\frac{5(a-c)^2}{48b}, \underline{\frac{5(a-c)^2}{32b}}$
Compete in Quantities	$\underline{\frac{5(a-c)^2}{32b}}, \frac{5(a-c)^2}{48b}$	$\underline{\frac{(a-c)^2}{9b}}, \underline{\frac{(a-c)^2}{9b}}$

- Conditional on firm 2 participating in the cartel, firm 1 compares $\frac{(a-c)^2}{8b} < \frac{5(a-c)^2}{36b} \iff 0.125 < 0.1388$.
- Conditional on firm 2 competing in quantities, firm 1 compares $\frac{5(a-c)^2}{48b} < \frac{(a-c)^2}{9b} \iff 0.1 < 0.111$.
- (And similarly for firm 2).

- Hence, deviating to Cournot output levels is a best response for every firm regardless of whether its rival respects or violates the cartel agreement.
- In other words, deviating to Cournot output levels is a strictly dominant strategy for both firms, and thus constitutes the NE of this game.
- How can firms then collide effectively? By interacting for several periods. (We will come back to collusive practices in future chapters).

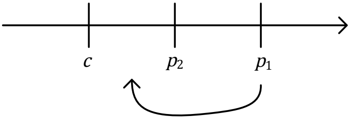
Bertrand Game of Price Competition

Competition in prices. The firm with the lowest price attracts all consumers. If both firms charge the same price, they share consumers equally.

- Any $p_i < c$ is strictly dominated by $p_i \geq c$.
- No *asymmetric* Nash equilibrium: (See Figures)
    1. If $p_1 > p_2 > c$, then firm 1 obtains no profit, and it can undercut firm 2's price to $p_2 > p_1 > c$. Hence, there exists a profitable deviation, which shows that $p_1 > p_2 > c$ cannot be a psNE.
    2. If $p_2 > p_1 > c$. Similarly, firm 2 obtains no profit, but can undercut firm 1's price to $p_1 > p_2 > c$. Hence, there exists a profitable deviation, showing that $p_2 > p_1 > c$ cannot be a psNE.
    3. If $p_1 > p_2 = c$, then firm 2 would want to raise its price (keeping it below p_1). Hence, there is a profitable deviation for firm 2, and $p_1 > p_2 = c$ cannot be a psNE.
    4. Similarly for $p_2 > p_1 = c$.

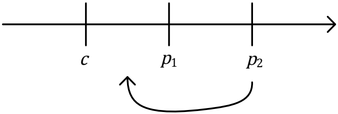
Bertrand Game of Price Competition

1. $p_1 > p_2 > c$



Profitable deviation of firm 1.

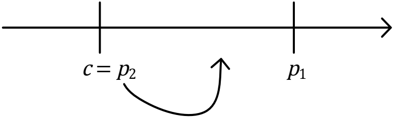
2. $p_2 > p_1 > c$



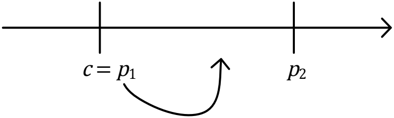
Profitable deviation of firm 2.

Bertrand Game of Price Competition

1. $p_1 > p_2 = c$

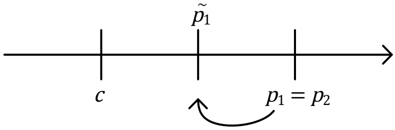


2. $p_2 > p_1 = c$



Bertrand Game of Price Competition

- Therefore, it must be that the psNE is symmetric. If $p_1 = p_2 > c$, then both firms have incentives to deviate, undercutting each other's price (keeping it above c, e.g., $p_2 > \tilde{p}_1 > c$.



- Hence, $p_1 = p_2 = c$ is the unique psNE.

Bertrand Game of Price Competition

- The Bertrand model of price competition predicts intense competitive pressures until both firms set prices $p_1 = p_2 = c$.
- How can the "super-competitive" outcome where $p_1 = p_2 = c$ be ameliorated? Two ways:
  - Offering price-matching guarantees.
  - Product differentiaion

Price Competition with Differentiated Products

- Another variation of the standard Bertrand model of price competition is to allow for product differentiation:
  - In the standard Bertrand model, firms sell a homogeneous (undifferentiated) product, e.g., wheat.
  - We will now see what happens if firms sell heterogeneous (differentiated) products, e.g., Coke and Pepsi.
- Let's consider the following example from Harrington (pp. 160-164) analyzing the competition between Dell and HP.

Price Competition with Differentiated Products

- Demand for Dell computers

$$q_{Dell}(p_{Dell}, p_{HP}) = 100 - 2p_{Dell} + p_{HP}$$

so that an increase in p_{Dell} *reduces* the demand for Dell computers (own-price effect), but an increase in p_{HP} actually *increases* the demand for Dell computers (cross-price effect).

- Similarly for HP,

$$q_{HP}(p_{HP}, p_{Dell}) = 100 - 2p_{HP} + p_{Dell}$$

- Hence, profits for Dell are

$$\pi_{Dell}(p_{Dell}, p_{HP}) = \underbrace{[p_{Dell} - 10]}_{\text{Profits per unit}} \underbrace{(100 - 2p_{Dell} + p_{HP})}_{q_{Dell}}$$

or, expanding it,

$$100p_{Dell} - 2p_{Dell}^2 + p_{HP}p_{Dell} - 100 + 20p_{Dell} - 10p_{HP}$$

Price Competition with Differentiated Products

- Taking FOCs with respect to p_{Dell} (the only choice variable for Dell), we obtain

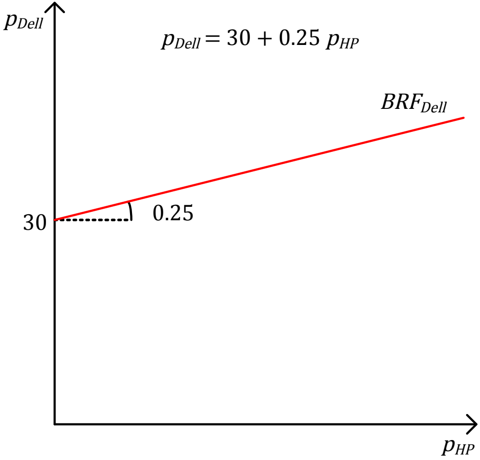
$$\frac{\partial \pi_{Dell}(p_{Dell}, p_{HP})}{\partial p_{Dell}} = 100 - 4p_{Dell} + p_{HP} + 20 = 0$$

  and solving for p_{Dell} we find

$$p_{Dell} = \frac{120 + p_{HP}}{4} = 30 + 0.25p_{HP} \qquad (BRF_{Dell})$$

- (See figure).

Price Competition with Differentiated Products



- Note the difference with the Cournot model of price competition: *BRF* is positively (not negatively) sloped.
  - *Intuition*: strategic complementarity vs. strategic substitutability.

Price Competition with Differentiated Products

- Similarly operating with HP (where marginal costs are $c = 30$), we have

$$\pi_{HP}(p_{HP}, p_{Dell}) = \underbrace{[p_{HP} - 30]}_{\text{Profits per unit}} \underbrace{(100 - 2p_{HP} + p_{Dell})}_{q_{HP}}$$

- Taking FOCs with respect to p_{HP} (the only choice variable for HP), we obtain

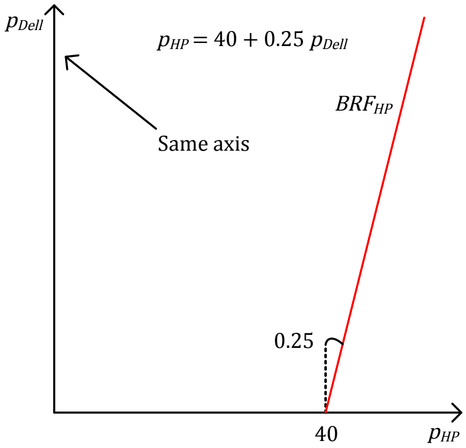
$$\frac{\partial \pi_{HP}(p_{HP}, p_{Dell})}{\partial p_{HP}} = 100 - 4p_{HP} + p_{Dell} + 60 = 0$$

  and solving for p_{HP} we find

$$p_{HP} = \frac{160 + p_{Dell}}{4} = 40 + 0.25p_{Dell} \qquad (BRF_{HP})$$

- (See figure).

Price Competition with Differentiated Products



Price Competition with Differentiated Products

- As a side, note that the SOCs for a max are also satisfied since:

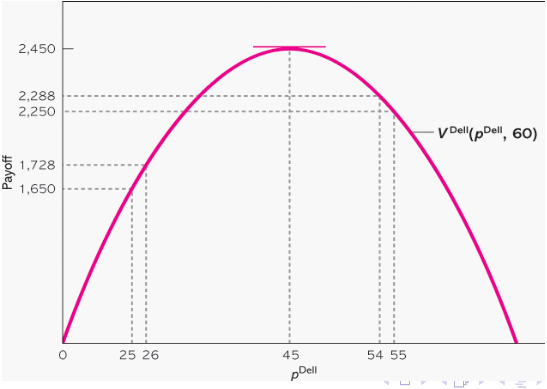
$$\frac{\partial^2 \pi_{Dell}(p_{Dell}, p_{HP})}{\partial p_{Dell}^2} = -4 < 0 \text{ (Dell's profit function is concave)},$$
$$\frac{\partial^2 \pi_{HP}(p_{HP}, p_{Dell})}{\partial p_{HP}^2} = -4 < 0 \text{ (HP's profit function is concave)}$$

Price Competition with Differentiated Products

- Indeed, if we graphically represent Dell's profit function for $p_{HP} = \$60$, that is

$$(p_{Dell} - 10)(100 - 2p_{Dell} + 60) = 180p - 2p^2 - 1600$$

we obtain the following concave profit function:



Price Competition with Differentiated Products

- Hence, both firms' *BRF*s cross at

$$p_{Dell} = 30 + 0.25 \underbrace{\left(40 + 0.25 p_{Dell}\right)}_{p_{HP}} = 30 + 10 + 0.625 p_{Dell}$$

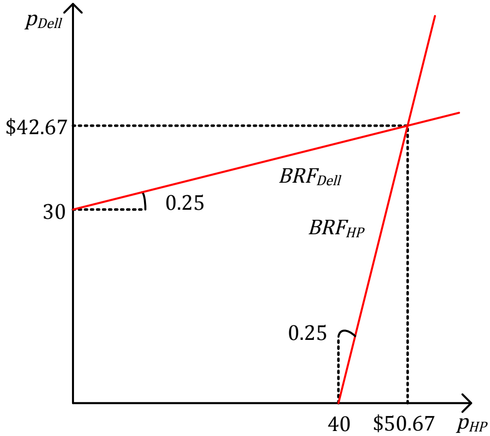
  and solving for p_{Dell} (the only unknown), we obtain $p_{Dell} = \$42.67$.

- We can now find p_{HP} by just plugging $p_{Dell} = \$42.67$ into BRF_{HP}, as follows

$$p_{HP} = 40 + 0.25 p_{Dell} = 40.25 + 0.25 * 42.67 = \$50.67$$

Price Competition with Differentiated Products

- Putting BRF_{Dell} and BRF_{HP} together



More Problems that Include Continuum Strategy Spaces

- Let's move outside the realm of industrial organization. There are still several games where players select an action among a continuum of possible actions.
- What's ahead...
- **Tragedy of the commons:** how much effort to exert in fishing, exploiting a forest, etc, incentives to overexploit the resource.
- **Tariff setting by two countries:** what precise tariff to set.
- **Charitable giving:** how many dollars to give to charity.
- **Electoral competition:** political candidates locate their platforms along the line (left-right, more or less spending, more or less security, etc.)
- **Accident law:** how much care a victim and an injurer exert, given different legal rules.