Chapter 2: Equilibirum dominance

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Introduction

Strictly Dominated Strategy

- Strategy provides a player with a strictly lower payout than another available strategy, regardless of the opponent's strategy
- Player i finds that a strategy s_i strictly dominates strategy $s_i^{'} \neq s_i$ if

$$u_i(s_i, s_{-i}) > u_i(s_i^{'}, s_{-i})$$

Iterative deleting strategies in such way is called
Iterated Deletion of Strictly Dominated Strategies, (IDSDS).

How to find strictly dominated strategies

1. Fix your attention on one strategy of the column player, s_2 (i.e., one specific column). Find a strategy $s_1^{'}$ that yields a strictly lower payoff than some other strategy s_1 for the row player, that is, $u_1(s_1, s_2) > u_1(s_1^{'}, s_2)$ for a given strategy of player 2.
2. Repeat step 1, but now fix your attention on a different column. That is, and if the above payoff inequality also holds, $u_1(s_1, s_2^{'}) > u_1(s_1^{'}, s_2^{'})$, which is now evaluated at a different strategy of player 2.
3. If, after repeating step 1 for all possible strategies of player 2 (all columns), you and that strategy $s_1^{'}$ yields a strictly lower payoff for player 1 than strategy s_1, you can claim that strategy $s_1^{'}$ is strictly dominated by s_1.

Otherwise, player 1 does not have a strictly dominated strategy

Example: strictly dominated strategies

Firm 2

		h	*m*	*l*
	H	2, 2	3, 1	5, 0
Firm 1	*M*	1, 3	2, 2	2.5, 2.5
	L	0, 5	2.5, 2.5	3, 3

Matrix 2.1. Strictly dominated strategies in the output game.

Applying IDSDS

1. From the definition of rationality, we know that a player would never use strictly dominated strategies, so we can delete them from her original strategy set, S_i, obtaining S_i' (which is technically a subset of S_i).
2. We can then proceed by also using common knowledge of rationality, which entails that every player j can put in her opponent's shoes, identify all strictly dominated strategies for her opponent, and delete them.
3. Continue again: player i considers now her rival's reduced strategy set S_j' and finds whether some of his own strategies in $S_i' \times S_j'$ now become strictly dominated. At the end of this step, we obtain a further reduced strategy set S_i''.

Example: IDSDS

		Firm 2	
		h	l
	H	4, 4	0, 2
Firm 1	M	1, 4	2, 0
	L	0, 2	0, 0

Matrix 2.2a. When IDSDS yields a unique equilibrium.

IDSDS Properties

1. Does order of deletion matter in IDSDS?
   - order of deletion does not matter when applying IDSDS
   - We would predict different equilibrium results if we started deleting strictly dominated strategies for player 1 or 2
2. Deleting more than one strategy at a time
   - We can delete all strictly dominated strategies we found for that player, so we do not need to delete one strictly dominated strategy at a time, saving us valuable time.
3. Multiple equilibrium predictions
   - use of IDSDS does not necessarily provide such a precise prediction, but rather two or more strategy profiles

IDSDS: Multiple Equilibria

Firm 2

		h	*m*	*l*
Firm 1	*H*	2, 3	1, 4	3, 2
	M	5, 3	2, 1	1, 2
	L	3, 6	4, 7	5, 4

Matrix 2.3a. When IDSDS yields more than one equilibrium-I.

Firm 2

		h	*m*	*l*
Firm 1	*M*	5, 3	2, 1	1, 2
	L	3, 6	4, 7	5, 4

Matrix 2.3b. When IDSDS yields more than one equilibrium-II.

Firm 2

		h	*m*
Firm 1	*M*	5, 3	2, 1
	L	3, 6	4, 7

Matrix 2.3c. When IDSDS yields more than one equilibrium-III.

IDSDS: Prisoner's Dilemma

		Player 2	
		Confess	Not confess
Player 1	Confess	$-4, -4$	$0, -8$
	Not confess	$-8, 0$	$-2, -2$

Figure: Matrix 2.4a: The Prisoner's Dilemma game.

IDSDS: Prisoner's Dilemma

		Player 2	
		Confess	Not confess
Player 1	Confess	$-4, -4$	$0, -8$
	Not confess	$-8, 0$	$-2, -2$

Figure: Matrix 2.4a: The Prisoner's Dilemma game.

		Player 2	
		Confess	Not confess
Player 1	Confess	$-4, -4$	$0, -8$

Matrix 2.4b. The Prisoner's Dilemma game after one round of IDSDS.

IDSDS: Prisoner's Dilemma

		Player 2 Confess	Not confess
Player 1	Confess	$-4, -4$	$0, -8$
	Not confess	$-8, 0$	$-2, -2$

Figure: Matrix 2.4a: The Prisoner's Dilemma game.

		Player 2 Confess	Not confess
Player 1	Confess	$-4, -4$	$0, -8$

Matrix 2.4b. The Prisoner's Dilemma game after one round of IDSDS.

		Player 2 Confess
Player 1	Confess	$-4, -4$

Matrix 2.4c. The Prisoner's Dilemma game after two rounds of IDSDS.

Other Prisoner Dilemma Games

General form of Prisoner's Dilemma Games:

		Player 2 Confess	*Player 2* Not confess
Player 1	Confess	a, a	b, c
Player 1	Not confess	c, b	d, d

Matrix 2.4d. The Prisoner's Dilemma game - General form.

Coordination games - Battle of the Sexes (BOTS)

		Wife	
		Football, F	Opera, O
Husband	Football, F	$10, 8$	$6, 6$
	Opera, O	$4, 4$	$8, 10$

Matrix 2.5a. The Battle of the Sexes game.

Coordination games - Battle of the Sexes (BOTS)

		Wife	
		Football, F	Opera, O
Husband	Football, F	$10, 8$	$6, 6$
	Opera, O	$4, 4$	$8, 10$

Matrix 2.5a. The Battle of the Sexes game.

$$IDSDS = \{(F, F), (F, O), (O, F), (O, O)\}$$

General form and Pareto Coordination Games

The general form of coordination games

		Wife	
		Football, F	Opera, O
Husband	Football, F	a_H, a_W	b_H, c_W
	Opera, O	c_H, b_W	d_H, d_W

Matrix 2.5b. Coordination games - General form.

General form and Pareto Coordination Games

The general form of coordination games

		Wife	
		Football, F	Opera, O
Husband	Football, F	a_H, a_W	b_H, c_W
	Opera, O	c_H, b_W	d_H, d_W

Matrix 2.5b. Coordination games - General form.

Pareto Coordination Game - Stag Hunt

		Player 2	
		Stag	*Hare*
Player 1	*Stag*	6, 6	1, 4
	Hare	4, 1	2, 2

Matrix 2.6a. The stag hunt game.

Anti Coordination Game - The Game of Chicken

		Player 2 Swerve	Stay
Player 1	Swerve	$-1, -1$	$-8, 10$
	Stay	$10, -8$	$-30, -30$

Matrix 2.7a. Anticoordination game.

		Player 2 Swerve	Stay
Player 1	Swerve	a, a	c, b
	Stay	b, c	d, d

Matrix 2.7b. Anticoordination game - General form.

Anti Coordination Game - The Game of Chicken

		Player 2: Swerve	Player 2: Stay
Player 1	Swerve	$-1, -1$	$-8, 10$
	Stay	$10, -8$	$-30, -30$

Matrix 2.7a. Anticoordination game.

		Player 2: Swerve	Player 2: Stay
Player 1	Swerve	a, a	c, b
	Stay	b, c	d, d

Matrix 2.7b. Anticoordination game - General form.

Movies with Anti-coordination games

- Rebel without a Cause (1955)
- Thirteen days (2000)
- Stand by Me (1986)
- A Beautiful Mind (2001)

Symmetric Games

Symmetric Game

- A two-player game is symmetric if both players' strategy sets coincide, $S_A = S_B$, and payoffs are unaffected by the identity of the player choosing each strategy, that is,

$$u_A(s_A, s_B) = u_B(s_A, s_B)$$

for every strategy profiles (s_A, s_B)

- Property is also known as payoff function satisfies *anonymity*: Starting from a setting where player A chooses s_A and player B chooses s_B, if we were to switch the identities of players A and B, so that player A becomes B and player B becomes A, every player's payoff would be unaffected.
- Examples: Prisoners' Dilemma, Game of Chicken

Asymmetric Games

Asymmetric Game

- A two-player game is asymmetric if players face different strategy sets, $S_A \neq S_B$; or if, despite facing the same strategy sets, $S_A = S_B$, their payoffs satisfy

$$u_A(s_A, s_B) \neq u_B(s_A, s_B)$$

  for atleast one strategy profile (s_A, s_B)
- Example: Battle of the Sexes

Randomization to bring IDSDS Further

Sometimes IDSDS has little to no "bite" in the game. When we allow players to randomize between two or more strategies, IDSDS might get some "bite" back

Firm 2

Firm 1			h	m	l
		H	0, 10	4, 6	4, 6
	Prob. $(1-q) \rightarrow$	M	4, 6	0, 10	6, 4
	Prob. $q \rightarrow$	L	10, 0	6, 4	4, 6

Matrix 2.8. Applying IDSDS and allowing for randomizations-I.

Randomization I

Let $\sigma = qL + (1 - q)M$

$$\begin{aligned} EU_1(\sigma|h) &= q10 + 4(1-q) = 4 + 6q \\ EU_1(\sigma|m) &= q6 + 0(1-q) = 6q \\ EU_1(\sigma|l) &= q4 + 6(1-q) = 6 - 2q \end{aligned}$$

Figure: Expected Utility from assigning a probability weight q

Inserting them into Matrix 2.8

		Firm 2		
		h	m	l
Firm 1	H	0	4	4
	$qL + (1-q)M$	$4 + 6q$	$6q$	$6 - 2q$

Matrix 2.9. Applying IDSDS and allowing for randomization-II.

Randomization II

Firm 1's expected payoff need to satisfy:

4+6q>0 or q>-2/3 when firm 2 chooses *h*

6q>4 or q>2/3 when firm 2 chooses *m*, and

6-2q>4 or q<1 when firm 2 chooses *l*

First and last condition on probability q hold because q must satisfy $q \in (0, 1)$ by assumption. Second condition, $q > 2/3$ restricts the range of probability weights we can use in randomization $qL + (1 - q)M$.

		Firm 2		
		h	m	l
Firm 1	M	4, 6	0, 10	6, 4
	L	10, 0	6, 4	4, 6

Matrix 2.10. Applying IDSDS and allowing for randomizations-III.

Randomization III

We can move to Firm 2 to note that *h* is strictly dominated by *m* since *h* yields a strictly lower payoff than m does, both when firm 1 chooses *M* in the top row

Randomization III

We can move to Firm 2 to note that h is strictly dominated by m since h yields a strictly lower payoff than m does, both when firm 1 chooses M in the top row

		Firm 2	
		m	l
Firm 1	M	0, 10	6, 4
	L	6, 4	4, 6

Matrix 2.11. Applying IDSDS and allowing for randomizations-IV.

What if IDSDS has no bite?

In some games,

- Players may have no strictly dominated strategies that we can delete using IDSDS
- which is informally described as that IDSDS "has no bite"

What to do?

- Apply Nash Equilibrium!

Evaluating IDSDS as a concept

1. Exsistence? Yes.
   - When we apply IDSDS to any game, we find that at least one equilibrium exists.
2. Uniqueness? No.
   - IDSDS does not necessarily provide a unique equilibrium outcome for all games since more than one cell may survive IDSDS
3. Robust to small payoff perturbations? Yes.
   - If we change the payoff of one of the players by a small amount, IDSDS still yields the same equilibrium outcome/s
4. Socially optimal? No.
   - The application of IDSDS does not necessarily yield socially optimal outcomes.
   - Prisoner's Dilemma game, where IDSDS provides us with a unique equilibrium prediction, (Confess, Confess), which does not coincide with the strategy profile that maximizes the sum of players' payoffs, (Not Confess, Not Confess).

Weakly Dominated Strategies

Weakely dominated Strategies

- Player i finds that strategy s_i weakly dominates another strategy $s_i^{'}$ if:
  1. $u_i(s_i, s_{-i}) \geq u_i(s_i^{'}, s_{-i})$ for every strategy profile s_{-i} of player i's rivals, and
  2. $u_i(s_i, s_{-i}) > u_i(s_i^{'}, s_{-i})$ for at least one strategy profile s_{-i}

		Firm 2		
		h	m	l
	H	2, 2	3, 2	5, 0
Firm 1	M	2, 3	3, 2	2.5, 2.5
	L	0, 5	2.5, 2.5	3, 3

Matrix 2.12. Finding weakly dominated strategies.

Does weakely dominated mean strictly dominated?

$$s_i \text{ is strictly dominated } \underset{\not\Leftarrow}{\Longrightarrow} s_i \text{ is weakly dominated}$$

- In Matrix 2.12, strategy *L* is strictly dominated by *H*, since *L* yields an unambiguously lower payoff for firm 1 than *H* does, regardless of the column that firm 1 chooses.
- This means that requirement 2 of the weak dominance definition holds for all firm 2's strategies, meaning that *L* is also weakly dominated by *H*
- However, strategy *M* is weakly dominated by *H* but *M* is not strictly dominated by *H*

Deletion order matters in IDWDS

- IDWDS may yield different equilibrium predictions depending on which player we start with

		Firm 2		
		h	m	l
Firm 1	H	$2, 2$	$3, 2$	$5, 0$
	L	$0, 5$	$2.5, 2.5$	$3, 3$

Matrix 2.13a. Applying IDWDS starting with firm 1 ⊦ First step.

Deletion order matters in IDWDS

- IDWDS may yield different equilibrium predictions depending on which player we start with

		Firm 2		
		h	m	l
Firm 1	H	$2, 2$	$3, 2$	$5, 0$
	L	$0, 5$	$2.5, 2.5$	$3, 3$

Matrix 2.13a. Applying IDWDS starting with firm 1 ‐ First step.

Starting with deleting weakly dominated strategies for firm 1

$$IDWDS = \{(H, h), (H, m)\}$$

Deletion order matters in IDWDS

- IDWDS may yield different equilibrium predictions depending on which player we start with

		Firm 2		
		h	m	l
Firm 1	H	$2, 2$	$3, 2$	$5, 0$
	L	$0, 5$	$2.5, 2.5$	$3, 3$

Matrix 2.13a. Applying IDWDS starting with firm 1 ├ First step.

Starting with deleting weakly dominated strategies for firm 1

$$IDWDS = \{(H, h), (H, m)\}$$

Starting with deleting weakly dominated strategies for firm 2

$$IDWDS = \{(H, h), (M, h)\}$$

IDSDS vs. IDWDS

We say that the set of strategy profiles surviving IDWDS is a subset of those surviving IDSDS

$$\text{IDWDS} \implies \text{IDSDS}$$
$$\not\Leftarrow$$

Example

		Firm 2 h	*Firm 2* m
Firm 1	H	$2, 2$	$3, 2$
Firm 1	M	$2, 3$	$3, 2$

Matrix 2.15. Applying IDSDS in matrix 2.11.

$$IDSDS = \{(H, h), (H, m), (M, h), (M, m)\}$$

Strictly Dominant Strategy and Strictly dominant equlibirum (SDE)

Strictly Dominant Strategy

- Player i finds that strategy s_i is strictly dominant if

$$u_i(s_i, s_{-i}) > u_i(s_i^{'}, s_{-i})$$

for every strategy $s_i^{'} \neq s_i$ and every strategy profile s_{-i} of player i's rivals.

Strictly Dominant Equilibrium (SDE)

- A strategy profile $s^{SD} = (s_i^{SD}, s_{-i}^{SD})$ is a strictly dominant equilibrium (SDE) if every player i finds her strategy, s_i^{SD} , to be strictly dominant.

Example SDEs

- Prisoner's Dilemma
  - *Confess* is a SDE since *Confess* is a strictly dominant strategy for every player
  - Intuitively, every player's payoff from *Confess* is higher than from Not confess regardless of what her opponent does
- Coordination and Anti-Coordination Games
  - Players do not have strictly dominant strategies and, hence, a SDE does not exist.

$$(s_i, s_{-i}) \text{ is an SDE } \implies (s_i, s_{-i}) \text{ survives IDSDS.}$$

$$\not\Leftarrow$$

Evaluating SDE as a solution concept

1. Existence? No.
   - When we seek to find strictly dominant strategies for each player, in our search of a SDE, we may find games where one or more players do not have a strictly dominant strategy, implying that an SDE does not exist
2. Uniqueness? Yes.
   - While a SDE may not exist in some games, when we nd one, it must be unique.
3. Robustness to small payoff perturbation? Yes.
   - If we change the payoff of one of the players by a small amount ($\epsilon \to 0$), SDE still yields the same equilibrium prediction.
4. Socially optimal? No.
   - SDE of a game does not need to be socially optimal.
   - Example: Prisoner's Dilemma game, the SDE is (Confess, Confess), which does not coincide with the strategy profile that maximizes the sum of players' payoffs