Introduction to Game Theory

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- Game Theory as the study of interdependence
  - "No man is an island"
- Definition:
  - Game Theory: a formal way to analyze **interaction** among a **group** of **rational** agents who behave strategically.

- Several important elemants of this definition help us understand what is game theory, and what is not:
- **Interaction:** If your actions do not affect anybody else, that is not a situation of interdependence.
- **Group:** we are not interested in games you play with your imaginary friend, but with other people, firms, etc.
- **Rational agents:** we assume that agents will behave rationally especially if the stakes are high and you allow them sufficient time to think about their available strategies.
    - Although we mention some experiments in which individuals do not behave in a completely rational manner...
    - these "anomalies" tend to vanish as long as you allow for sufficient repetitions, i.e., everybody ends up learning, or you raise stakes sufficiently (high incentives).

Examples (1):

- Output decision of two competing firms:
  - Cournot model of output competition.
- Research and Development expenditures:
  - They serve as a way to improve a firm's competitiveness in posterior periods.
- OPEC pricing, how to sustain collusion in the long run...

Examples (2):

- Sustainable use of natural resources *and* overexploitation of the common resource.
- Use of environmental policy as a policy to promote exports.
  - Setting tax emission fees in order to favor domestic firms.
- Public goods (everybody wants to be a "free-rider").
  - I have never played a public good game!
  - Are you sure? A group project in class. The slacker you surely faced was our "free-rider."

Rules of a General Game (informal):(WATSON CH.2,3)

The rules of a game seek to answer the following questions:

1. Who is playing ?$\leftarrow$ set of players (I)
2. What are they playing with ?$\leftarrow$ Set of available actions (S)
3. Where each player gets to play ?$\leftarrow$ Order, or time structure of the game.
4. How much players can gain (or lose) ?$\leftarrow$ Payoffs (measured by a utility function $U_i(s_i, s_{-i})$

1. We assume **Common knowledge** about the rules of the game.

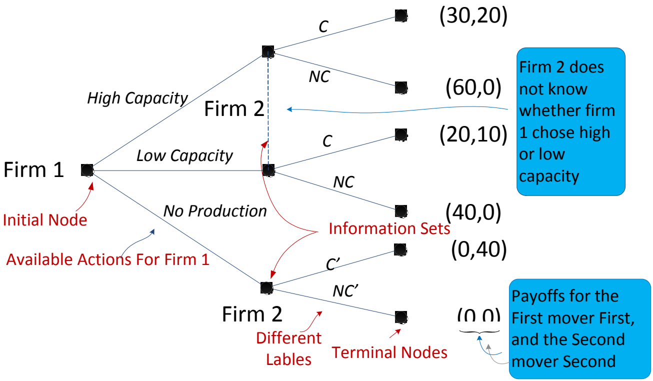
- As a player, I know the answer to the above four questions (rules of the game)
- In addition, I know that you know the rules, and...
- that you know that I know that you know the rules,.....(ad infinitum).

Two ways to graphically represent games

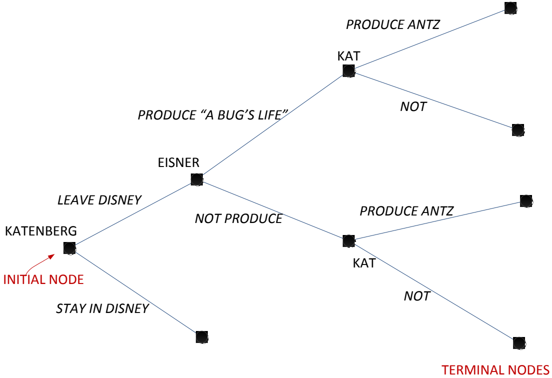
- Extensive form
  - We will use a game tree (next slide).
- Normal form (also referred as "strategic form").
  - We will use a matrix.

Example of a game tree

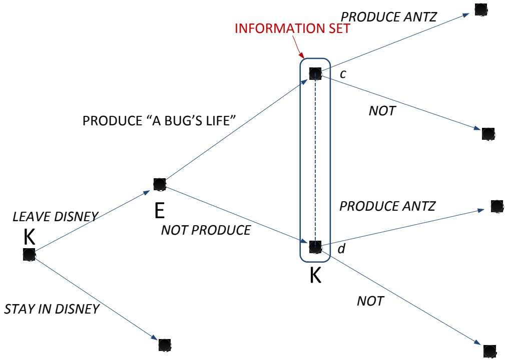
- Consider the following sequential-move game played by firms 1 and 2:
  - We will use a matrix



"ANTZ" vs. "A BUG'S LIFE"



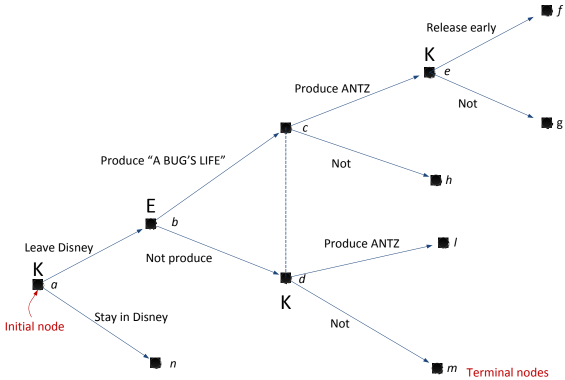
- In this example, Katsenberg observes whether Eisner produced the film "A BUG"S LIFE" or not before choosing to produce "ANTZ".



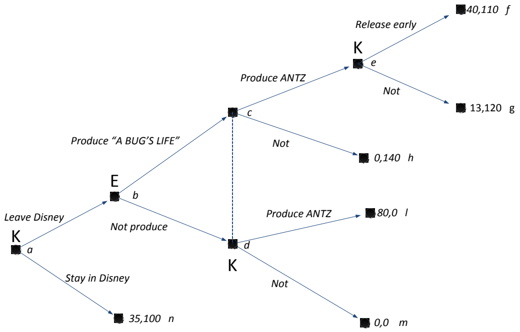
- When Katsenberg is at the move (either at node c or d), he knows that he is at one of these nodes,but he does not know at which one *and* the figure captures this lack of information with a dashed line connecting the nodes.:

The Bug Game

- We now add an additional stage at the end at which Katsenberg is allowed to release "Antz" early in case he produced the movie and Eisner also produced "A bug's life" (at node e).



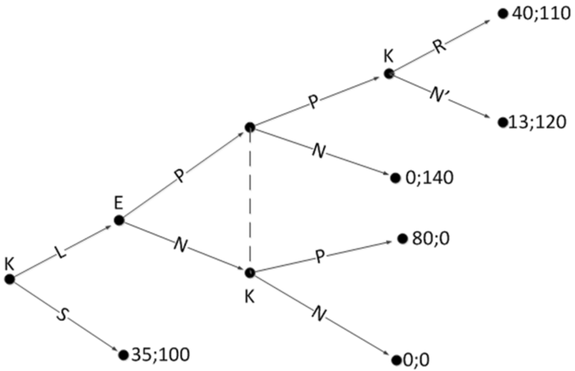
The Extensive Form of The Bug Game



- Let's define the payoff numbers as the profits that each obtains in the various outcomes, i.e.,in each terminal node.
- For example, in the event that Katzenberg stays at Disney, we assume he gets $35 million and Eisner gets $100 million (terminal node a).

The Bug Game Extensive Form (Abbreviating Labels)

- We often abbreviate labels in order to make the figure of the game tree less jammed, as we do next.



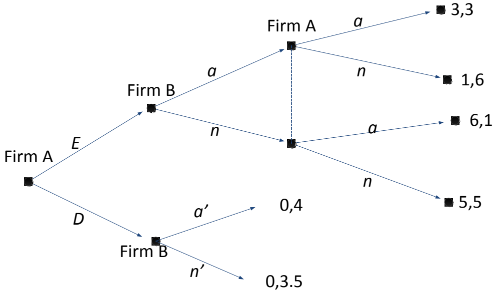
Information sets

- An information set is graphically represented with two or more nodes connected by a dashed line, (or a "sausage") including all these connected nodes.
- It represents that the player called to move at that information set cannot distinguish between the two or more actions chosen by his opponent before he is called to move.
- Hence, the set of available actions must be the same in all the nodes included on that information set (P and N in the previous game tree for Katsenberg).
  - Otherwise, Katsenberg, despite not observing Eisner's choice, would be able to infer it by analyzing which are the available actions he can choose from.

Guided exercise (page 19-20 in Watson)

- **Lets practice how to depict a game tree of a strategic situation on an industry:**
- Firm A decides whether to enter firm B's industry. Firm B observes this decision.
  - If firm A stays out, firm B alone decides whether to advertise. In this case, firm A obtains zero profits, and firm B obtains $4 million if it advertises and $3.5 million if it does not.
  - If firm A enters, both firms simultaneously decide whether to advertise, obtaining the following payoffs.
    - If both advertise, both firms earn $3 million.
    - If none of them advertise, both firms earn $5 million.
    - If only one firm advertises, then it earns $6 million and the other firm earns $1 million.

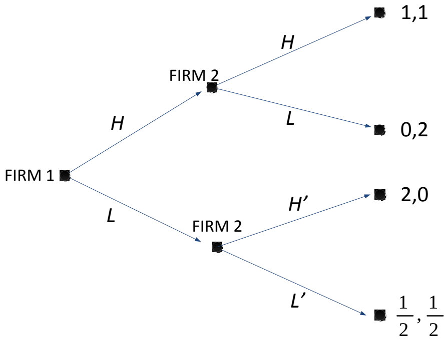
Guided Exercise, (continued)



- Let E and D denote firm A's initial alternatives of entering and not entering B's industry.
- Let a and n stand for "advertise" and "not advertise", respectively.
- Note that simultaneous advertising decisions are captured by firm A's information set.

Strategy: Definition of Strategy

- Lets practice finding the strategies of firm 1 and 2 in the following game tree:
  - We will use a matrix



Strategies for firm 1 : H and L.
Strategies for firm 2 : H. H';H. L';L. H;L

Strategy space and Strategy profile

- **Strategy space:** It is a set comprising each of the possible strategies of player i.
  - From our previous example:
    - $S_1 = \{H, L\}$ for firm 1
    - $S_2 = \{HH', HL', LH', LL'\}$ for firm 2.
- **Strategy profile**
  - It is a vector (or list) describing a particular strategy for every player in the game. For instance, in a two-player game

$$s = (s_1, s_2)$$

    where s_1 is a specific strategy for firm 1.(for instance, $s_1 = H$), and s_2 is a specific strategy for firm 2, e.g., $s_2 = LH'$.
  - More generally, for N players, a strategy profile is a vector with N components,

$$s = (s_1, s_2, s_3, ..., s_n)$$

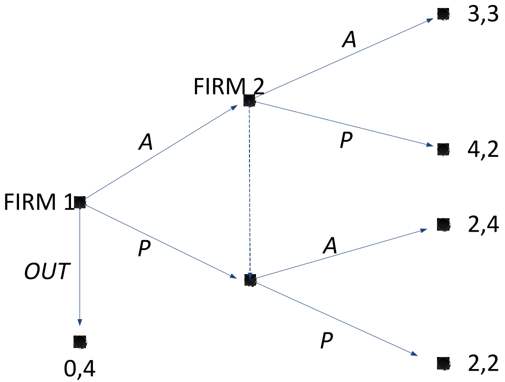
Strategy profile:

- In order to represent the strategies selected by all players except player i, we write:

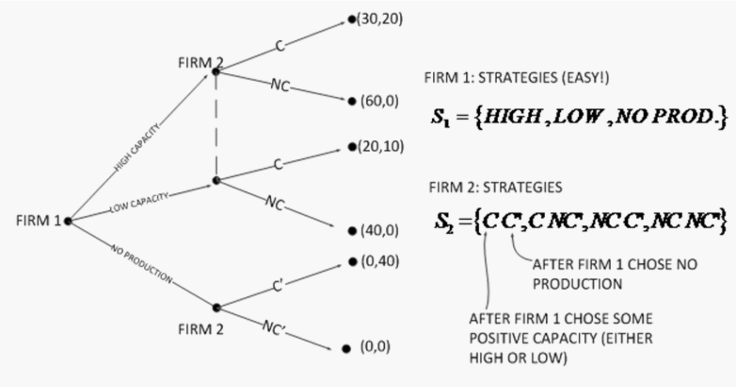
$$s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n)$$

  (Note that these strategies are potentially different)
- We can hence write, more compactly, as strategy profile with only two elements:
  The strategy player i selects, s_i, and the strategies chosen by everyone else, s_{-i}, as : $s = (s_i, s_{-i})$
- **Example:**
  - Consider a strategy profile s which states that player 1 selects B, player 2 chooses X, and player 3 selects Y, i.e., $s = (B, X, Y)$.Then,
    - $s_{-1} = (X, Y)$,
    - $s_{-2} = (B, Y)$, and
    - $s_{-3} = (B, X)$.

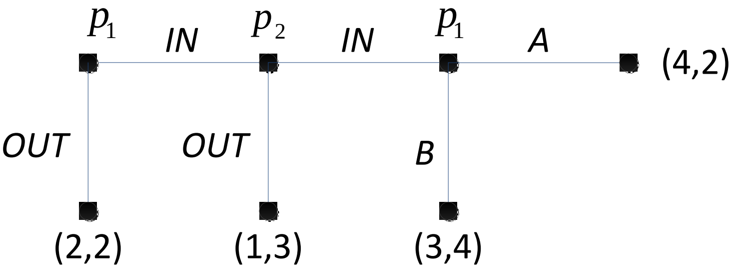
- Lets practice finding strategy sets in the following game tree:



- Let's define firm 1 and 2's available strategies in the first example of a game tree we described a few minutes ago:



ANOTHER EXAMPLE: THE CENTIPEDE GAME:



- Strategy set for player 2 : S_2 = {IN, OUT}
- Strategy set for player 1 : S_1 = {IN A, IN B, OUT A, OUT B}
- More examples on page 27 (Watson)

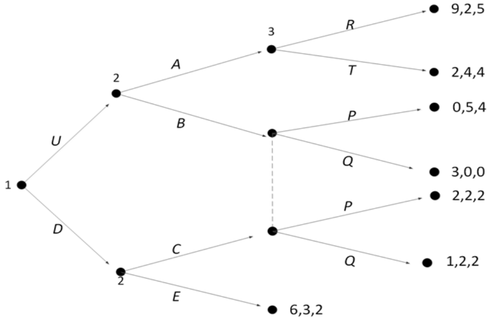
One second...

- Why do we have to specify my future actions after selecting "out" ? Two reasons:

1. Because of potential mistakes:
   - Imagine I ask you to act on my behalf, but I just inform you to select "out" at the initial node. However, you make a mistake (i.e., you play "In"), and player 2 responds with "In" as well. What would you do now??
   - With a strategy (complete contingent plan) you would know what to do even in events that are considered off the equillibrium path.
2. Because player 1's action later on affects player 2's actions, and ...
   - ultimately player 2's actions affects player 1's decision on whether to play "In" or "Out" at the beginning of the game.
   - This is related with the concept of backwards induction that we will discuss when solving sequential-move games.)

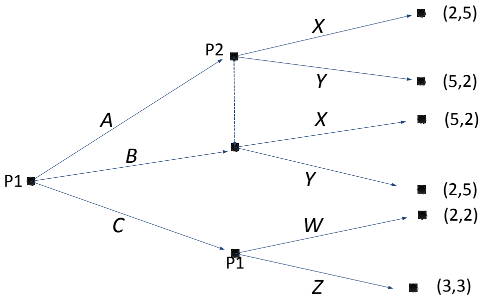
Some extensive-form games

- Let's now find the strategy spaces of a game with three players:



- S_1 = {U, D}
- S_2 = {AC, AE, BC, BE}; and
- S_3 ={RP,RQ,TP,TQ}

Some extensive-form games (Cont'l)



- $S_1 = \{AW, BW, CW, AZ, BZ, CZ\}$
- $S_2 = \{X, Y\}$

- **When a game is played simultaneously, we can represent it using a matrix**
  - *Example:* Prisoners' Dilemma game.

		Prisoner 2	
		Confess	Don't Confess
Prisoner 1	Confess	-5, -5	0, -15
	Don't Confess	-15, 0	-1, -1

- **Another example of a simultaneous-move game**
  - The "battle of the sexes" game.(I know the game is sexist, but please don't call it the "battle of the sexist" game !)

		Wife	
		Opera	Movie
Husband	Opera	1, 2	0, 0
	Movie	0, 0	2, 1

- **Yet, another example of a simultaneous-move game**
  - Pareto-coordination game.

		Firm 2	
		Superior tech.	Inferior tech.
Firm 1	Superior tech.	2, 2	0, 0
	Inferior tech.	0, 0	1, 1

- **Yet, another example of a simultaneous-move game**
  - The game of "chicken."

		Dean	
		Straight	Swerve
James	Straight	0, 0	3, 1
	Swerve	1, 3	2, 2

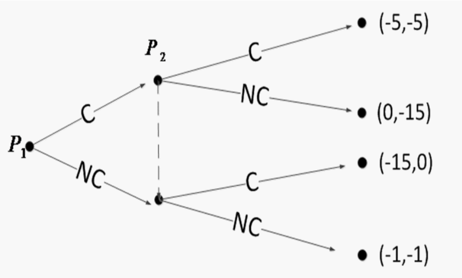
Other examples of the "Chickengame"

Mode	Description
Trackors	Footloose, (1984,Movie)
Bulldozers	Buster and Gob in Arrested Development (2004,TV)
Wheelchairs	Two old ladies with motorized wheelchairs in Banzai(2003,TV)
Snowmobiles	"[Two adult males] died in a head-on collision, earning a tie in the game of chicken they were playing with their snowmobiles" <www.seriouslyinternet.com/278.0.html>
Film Release Dates	Dreamworks and Disney-PIxar (2004)
Nuclear Weapons	Cuban Missile Crisis (1963)

Normal (Strategic) Form

- We can alternatively represent simultaneous-move games using a game tree, as long as we illustrate that players choose their actions without observing each others' moves, i.e., using information sets, as we do next for the prisoner's dilemma game:
- Extensive form representation of the Prisoner's Dilemma game :

		P_2	
		C	NC
P_1	C	-5,-5	0,-15
	NC	-15,0	-1,-1

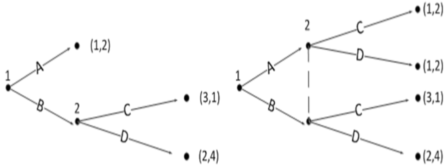


- **Practice** :Using a game tree, depict the equivalent extensive - form representation of the following matrix representing the "Battle of the Sexes" game.

		Wife	
		Opera	Movie
Husband	Opera	1,2	0,0
	Movie	0,0	2,1

Corresponding extensive and normal forms

- Only one way to go from extensive to normal form but potentially several ways to go from normal to extensive form, as the following example indicates.



		Player 2	
		C	D
Player 1	A	1,2	1,2
	B	3,1	2,4

- For this reason, we have to accurately describe which game we have in mind (the game tree in the left or right panel).

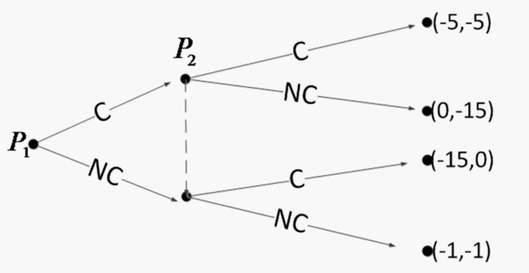
- **Additional practice?**
  - See "Guided Exercise" in page 34 of Watson.
  - This exercise transforms the Katsenberg-Eisner game into its matrix (normal form) representation.

von Neumann-Morgenstern expected utility function (WATSON CH.4)

- Expected utility (EU) that player i obtains from playing strategy s_i :

$$EU(s_i) = p_1 \cdot u(l_1) + p_2 \cdot u(l_2) + \ldots$$

  - Example:



- Let's consider that player 1 in the above game has a Bernoulli's utility function given by $u(I) = 3 \cdot I$,where I denotes income.
- Then, player 1 obtains the following expected utility from selecting C,

$$\begin{aligned} EU_1(C) &= prob(C) \cdot u((C, C)) + prob(NC) \cdot u((C, NC)) \\ &= p \cdot 3 \cdot (-5) + (1 - p) \cdot 0 = -15p \end{aligned}$$

where p represents the probability that player 2 chooses C.

- Similarly, player 1's expected utility from selecting NC is

$$\begin{aligned} EU_1(NC) &= prob(C) \cdot u((NC, C)) + prob(NC) \cdot u((NC, NC)) \\ &= p \cdot 3 \times (-15) + (1 - p) \cdot (3 \times (-1)) = 3 - 42p \end{aligned}$$

- In order to challenge ourselves a little bit further, let's find the expected utility that player 1 in the following game obtains when selecting U , C or D...
  - assuming that the probability with which his opponet, player 2, selects L, M and R are 1/2, 1/4 and 1/4 respectively.

PLAYER 2

		$\frac{1}{2}$L	$\frac{1}{4}$M	$\frac{1}{4}$R
	U	8,1	0,2	4,0
PLAYER 1	C	3,3	1,2	0,0
	D	5,0	2,3	8,1

- If player 1 believes player 2 will randomize according to probability distribution $\theta_2 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$,then player 1's expected utility is:

$$\begin{aligned} EU_1\left(U, \theta_2\right) &= \frac{1}{2} \cdot 8 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 4 = 5 \\ EU_1\left(C, \theta_2\right) &= \frac{1}{2} \cdot 3 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{7}{4} \\ EU_1\left(D, \theta_2\right) &= \frac{1}{2} \cdot 5 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 8 = 5 \end{aligned}$$

- What if player 2 believes player 1 will select $\theta_1 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ (U,C,D), and player 2 himself plans to randomize using $\left(0, \frac{1}{2}, \frac{1}{2}\right)$?
- Try on your own (answer in guided exercise, Ch4 Watson)

- **We are done discribing games!!**
  - We will return to some additional properties of game trees later on, but only for a second.
- **Let's start solving games!!**
  - We will use solution concepts that will help us predict the precise strategy that every player selects in the game.
- **Our goal:**
  - To be as precise as possible in our equilibrium predictions.
  - Hence, we will present (and rank) solution concepts in terms of their predictive power.