EconS 301- Intermediate Microeconomic Theory
Homework #5 - Due date: Thursday November 17th, 2022.

1. Consider a situation where two identical firms are simultaneously deciding whether to price high or price low. If both firms price high, they both receive half of the market's profits, $\frac{\pi}{2}$. If one firm prices high, while the other firm prices low, the low pricing firm receives all of the market's profits, π, while the high pricing firm receives 0 profits. If both firms price low, they both receive 0 profits.

   (a) Depict the normal form representation of this simultaneous move game. Find all psNE.

   - Based on the description of the game, the normal form representation is as follows:

Firm 1 \ *Firm 2*	High	Low
High	$\frac{\pi}{2}, \frac{\pi}{2}$	$0, \pi$
Low	$\pi, 0$	$0, 0$

   We can find the pure strategy Nash equilibria of this game by analyzing each firm's best responses.

   – *Firm 1's best responses:* When firm 2 chooses High, firm 1 can respond with High, leading to a payoff of $\frac{\pi}{2}$, or Low, leading to a payoff of π. Since $\frac{\pi}{2} < \pi$, firm 1's best response to firm 2 choosing High is Low, $BR_1(H) = L$. When firm 2 chooses Low, firm 1 can respond with High, leading to a payoff of 0, or Low, leading to a payoff of 0. Since $0 = 0$, firm 1 is indifferent to each of their options, so firm 1's best response to firm 2 choosing Low is High or Low, $BR_1(L) = \{H, L\}$.

Firm 1 \ *Firm 2*	High	Low
High	$\frac{\pi}{2}, \frac{\pi}{2}$	$\underline{0}, \pi$
Low	$\underline{\pi}, 0$	$\underline{0}, 0$

   – *Firm 2's best responses:* When firm 1 chooses High, firm 2 can respond with High, leading to a payoff of $\frac{\pi}{2}$, or Low, leading to a payoff of π. Since $\frac{\pi}{2} < \pi$, firm 2's best response to firm 1 choosing High is Low, $BR_2(H) = L$. When firm 1 chooses Low, firm 2 can respond with High, leading to a payoff of 0, or Low, leading to a payoff of 0. Since $0 = 0$, firm 2 is indifferent to each of their options, so firm 2's best response to firm 1 choosing Low is High or Low, $BR_1(L) = \{H, L\}$.

Firm 1 \ *Firm 2*	High	Low
High	$\frac{\pi}{2}, \frac{\pi}{2}$	$0, \underline{\pi}$
Low	$\pi, \underline{0}$	$0, \underline{0}$

- *Nash equilibrium*: With the best responses for each player identified, there exists three Nash equilibria in pure strategies, (H, L), (L, H), and (L, L). That being said, all Nash equilibria except for (L, L) are weakly dominated.

		Firm 2	
		High	Low
Firm 1	High	$\frac{\pi}{2}, \frac{\pi}{2}$	$\underline{0}, \underline{\pi}$
	Low	$\underline{\pi}, \underline{0}$	$\underline{0}, \underline{0}$

(b) Suppose now that the firms decided to collude to charge a high price. For what minimal discount factor δ do the firms cooperate by charging a high price?

- To analyze the repeated version of this game, we must examine the cheating history.
- *No previous cheating.* When we implement our Grim-Trigger strategy, as long as no cheating has occurred in the past, each firm continues to choose *High* earning a payoff of $\frac{\pi}{2}$ in every period.. This leads to the following stream of discounted payoffs,

$$\frac{\pi}{2} + \delta\left(\frac{\pi}{2}\right) + \delta^2\left(\frac{\pi}{2}\right) + ... = \frac{\pi}{2}(1 + \delta + \delta^2 + ...)$$

Once again, since $1 + \delta + \delta^2 + ... = \frac{1}{1-\delta}$, we can substitute to obtain,

$$\frac{\pi}{2}(1 + \delta + \delta^2 + ...) = \frac{1}{1-\delta}\left(\frac{\pi}{2}\right)$$

If one firm chose *Low* instead, they would receive a payoff of π immediately, then a payoff of 0 for every period after as both firms revert to choosing *Low* in every period. This leads to an alternative stream of discounted payoffs of,

$$\pi + \delta 0 + \delta^2 0 + ... = \pi$$

Either firm will prefer *High* over *Low* as long as their stream of discounted payoffs while choosing *High* are higher than that if they chose *Low*, i.e.,

$$\frac{1}{1-\delta}\left(\frac{\pi}{2}\right) \geq \pi$$

To solve this, first we multiply both sides of this inequality by $2(1-\delta)$, which gives us $\pi \geq 2\pi(1-\delta)$. Rearranging terms, we have $\pi \geq 2\pi - 2\pi\delta$. Reorganizing terms gives us $2\pi\delta \geq \pi$. Lastly, we divide both sides of this expression by 2π to obtain

$$\delta \geq \frac{1}{2}$$

which implies that any value of δ greater than $\frac{1}{2}$ can sustain cooperation in this case. By letting this inequality hold with equality, $\delta = \frac{1}{2}$, we have our minimal discount factor for firms to choose *High*. Interestingly, it does not even matter what the value of the profits are to the firms, as they completely cancel out.

- *Some cheating history.* Now we must examine if firms would want to deviate from their strategy of playing *Low* when someone has played *Low* in the past. By continuing to choose *Low*, each firm receives a payoff of 0 in each period, leading to the following stream of discounted payoffs,

$$0 + \delta 0 + \delta^2 0 + ... = 0$$

Alternatively, either firm could choose *High* for a period, earning 0 for that period and 0 for every period after as they revert back to choosing *Low*. This leads to an "alternative" stream of discounted payoffs of,

$$0 + \delta 0 + \delta^2 0 + ... = 0$$

Since deviating from choosing *Low* provides exactly the same discounted stream of payoffs as choosing *High*, this can be sustained for any value of δ.

- *spNE.* Thus, as long as $\delta \geq \frac{1}{3}$, our Grim-Trigger Strategy is a subgame perfect Nash equilibrium.

2. Calculate the HHI in the following markets, where three firms operate under different levels of market share:

(a) Each firm has an equal share of the market (i.e., 33.3 percent).

- The HHI is calculated as

$$\begin{aligned} HHI &= (33.3)^2 + (33.3)^2 + (33.3)^2 \\ &= 1,108.89 + 1,108.89 + 1,108.89 \\ &= 3,326.67. \end{aligned}$$

(b) One firm captures 50 percent of the market, while the other two each have 25 percent.

- The HHI is calculated as

$$\begin{aligned} HHI &= (50)^2 + (25)^2 + (25)^2 \\ &= 2,500 + 625 + 625 \\ &= 3,750. \end{aligned}$$

(c) One firm captures 80 percent of the market, while the other two each have 10 percent.

- The HHI is calculated as

$$\begin{aligned} HHI &= (80)^2 + (10)^2 + (10)^2 \\ &= 6,400 + 100 + 100 \\ &= 6,600. \end{aligned}$$

(d) Two firms have 45 percent of the market, while the other firm has 10 percent.

- The HHI is calculated as

$$\begin{aligned} HHI &= (45)^2 + (45)^2 + (10)^2 \\ &= 2,025 + 2,025 + 100 = \\ & \quad 4,150. \end{aligned}$$

(e) How do these different market shares (in parts a–d) affect the HHI?

- As more of the market share is concentrated in a single firm, the larger the HHI. The market described in part (c), where one firm has 80 percent of the market, the HHI is highest. Whereas when the market share is spread evenly between the three firms, as in part (a), HHI is at its lowest.

3. Consider a market with three firms producing a homogeneous good and facing a linear demand function $p(Q) = 1 - Q$, where $Q \equiv q_1 + q_2 + q_3$ denotes aggregate output. All firms face a constant marginal cost of production given by c, where $1 > c > 0$.

(a) Set up firm 1's PMP, differentiate with respect to its output q_1, and obtain this firm's best response function. [*Hint*: It should be a function of firm 2's and 3's output, q_2 and q_3.]

- Firm 1's profit maximization problem is

$$\max_{q_1} \ \pi_1 = (1 - q_1 - q_2 - q_3)q_1 - cq_1.$$

Differentiating with respect to q_1, we obtain

$$\frac{\partial \pi_1}{\partial q_1} = 1 - 2q_1 - q_2 - q_3 - c = 0,$$

rearranging, we find $2q_1 = 1 - q_2 - q_3 - c$, and solving for q_1, we obtain firm 1's best response function

$$q_1(q_2, q_3) = \frac{1-c}{2} - \frac{1}{2}(q_2 + q_3),$$

which originates at a vertical intercept of $\frac{1-c}{2}$, and decreases at a rate of $\frac{1}{2}$ when either firm 2 or 3 marginally increases its output.

(b) Repeat the process for firms 2 and 3, to obtain their best response functions. [*Hint*: You should find that all firms have symmetric best response functions.]

- *Firm 2.* Firm 2's profit maximization problem is

$$\max_{q_2} \pi_2 = (1 - q_1 - q_2 - q_3)q_2 - cq_2.$$

Differentiating with respect to q_2, we obtain

$$\frac{\partial \pi_2}{\partial q_2} = 1 - q_1 - 2q_2 - q_3 - c = 0,$$

rearranging, we find $2q_2 = 1 - q_1 - q_3 - c$, and solving for q_2, we obtain firm 2's best response function

$$q_2(q_1, q_3) = \frac{1-c}{2} - \frac{1}{2}(q_1 + q_3),$$

which is symmetric to firm 1's best response function. This comes at no surprise since all firms face the same inverse demand function and total cost function.

- *Firm 3.* Firm 3's profit maximization problem is

$$\max_{q_3} \pi_3 = (1 - q_1 - q_2 - q_3)q_3 - cq_3.$$

Differentiating with respect to q_3, we obtain

$$\frac{\partial \pi_3}{\partial q_3} = 1 - q_1 - q_2 - 2q_3 - c = 0,$$

rearranging, we find $2q_3 = 1 - q_1 - q_2 - c$, and solving for q_3, we obtain firm 3's best response function

$$q_3(q_1, q_2) = \frac{1-c}{2} - \frac{1}{2}(q_1 + q_2),$$

which is symmetric to firm 1's best response function. Again, this is not surprising since all firms face the same inverse demand function and total cost function.

(c) Interpret firm 1's best response function: if firm 2 were to marginally increase its output, does firm 1 increase or decrease its own output? Either way, by how much?

- To find this, we differentiate firm 1's best response function with respect to firm 2's output q_2:

$$\frac{\partial q_1(q_2, q_3)}{\partial q_2} = -\frac{1}{2}.$$

For each unit increase in firm 2's output, firm 1 decreases its output by a half unit.

(d) Using the three best response functions for these firms, find the point where they cross. The triplet (q_1^*, q_2^*, q_3^*) characterizes the NE of this Cournot game.

- Since all firms are symmetric, a symmetric equilibrium must exist where all firms produce the same output level $q_1^* = q_2^* = q_3^* = q^*$. We can plug this property into firm 2's to obtain

$$q^* = \frac{1-c}{2} - \frac{1}{2}\left(q^* + q^*\right),$$

which is equivalent to dropping the subscripts of all output levels. Simplifying, we find

$$2q^* = 1 - c - 2q^*,$$

or $4q^* = 1 - c$. Solving for q^*, we find that every firm produces an equilibrium output of

$$q^* = \frac{1-c}{4}.$$

(e) Is the equilibrium output that you found in part (d) increasing or decreasing in marginal cost c?

- Differentiating the equilibrium output with respect to c, we obtain

$$\frac{\partial q^*}{\partial c} = -\frac{1}{4} < 0.$$

Therefore, as marginal cost increases, the equilibrium output for each firm decreases.

(f) Find the price that emerges in equilibrium, along with the profits that every firm earns.

- The price each firm faces is

$$\begin{aligned} p &= 1 - q_1 - q_2 - q_3 \\ &= 1 - 3q^* \\ &= 1 - 3\frac{(1-c)}{4} \\ &= \frac{1+3c}{4}. \end{aligned}$$

- Each firm earns the same profit in equilibrium; that is,

$$\pi = pq - cq,$$

or, plugging in for p^* and q^*,

$$\begin{aligned} \pi &= \left(\frac{1+3c}{4}\right)\frac{1-c}{4} - c\frac{1-c}{4} \\ &= \frac{(1-c)^2}{16}, \end{aligned}$$

which coincides with the square of the individual output in equilibrium, $\pi = (q^*)^2$.

4. Two gasoline stations are situated across the street from each other and are in fierce competition. They face market demand of $p = 10 - 0.05Q$, where $Q = q_1 + q_2$ denotes aggregate output, and each has total cost $TC(q_i) = 10 + 0.5q_i$, where $i \in \{1, 2\}$ denotes the firm.

(a) If firms compete in quantities, find each firm's best response function.

- Gas station 1's profit-maximization problem is

$$\max_{q_1} \ \pi_1 = [10 - 0.05(q_1 + q_2)]\, q_1 - (10 + 0.5q_1)$$

Differentiating with respect to q_1, we obtain

$$\frac{\partial \pi_1}{\partial q_1} = 10 - 0.1q_1 - 0.05q_2 - 0.5 = 0$$

Rearranging, $0.1q_1 = 9.5 - 0.05q_2$, and solving for q_1 we have firm 1's best response function

$$q_1(q_2) = 95 - \frac{1}{2}q_2$$

which originates at a vertical intercept of 95 units and decreases at a rate of $\frac{1}{2}$ for each unit of output from its rival.

- The other gas station has a symmetric best response function, yielding

$$q_2(q_1) = 95 - \frac{1}{2}q_1$$

(b) Find equilibrium output for each firm, price, and profits.

- In a symmetric equilibrium, both gas stations produce the same output, $q_1 = q = q$, so the above best response function becomes

$$q = 95 - \frac{1}{2}q$$

or, after rearranging, $\frac{3}{2}q = 95$, which solving for q yields an equilibrium output of

$$q = 95\frac{2}{3} = 63.33 \text{ units.}$$

- Plugging this output level into the inverse demand, we find the equilibrium price

$$p^* = 10 - 0.05(63.33 + 63.33) = \$3.67$$

- Each firm earns an equilibrium profit of

$$\pi_i = 3.67(63.33) - (10 + 0.5(63.33)) = \$190.76$$

(c) If firms collude, what equilibrium price and quantity will each firm offer? What will their profits be?

- If the firms collude, they will maximize their joint profit, as follows

$$\begin{aligned}\max_{q_1,q_2} \ \pi &= \pi_1 + \pi_2 \\ &= \underbrace{[10 - 0.05(q_1+q_2)]\, q_1 - (10 + 0.5q_1)}_{\pi_1} + \underbrace{[10 - 0.05(q_1+q_2)]\, q_2 - (10 + 0.5q_2)}_{\pi_2}\end{aligned}$$

Combining terms, we obtain

$$\max_{q_1,q_2} \ [10 - 0.05(q_1+q_2)]\,(q_1+q_2) - 20 - 0.5(q_1+q_2)$$

Since $Q = q_1 + q_2$ denotes aggregate output, the cartel's problem can be expressed as

$$\max_{Q} \ (10 - 0.05Q)Q - 20 - 0.5Q$$

Intuitively, the cartel chooses the aggregate output Q that maximizes the joint profits of both firms. Differentiating with respect to Q, we find

$$10 - 0.1Q - 0.5 = 0$$

rearranging, $0.1Q = 9.5$, and solving for Q,

$$Q = 95 \text{ units.}$$

Since firms are symmetric, each firm produces half of this aggregate output, or

$$q_1 = q_2 = \frac{95}{2} = 47.5 \text{ units.}$$

- Plugging $Q = 95$ into the inverse demand, the cartel price is

$$p^C = 10 - 0.05(95) = \$5.25$$

- Cartel overall profits are

$$\pi = (5.25)95 - 20 - 0.5(95) = \$431.22$$

The two gas stations evenly split this profit, earning $\frac{431.22}{2} = \$215.63$ each.

(d) If the firms play an infinitely repeated game, and they seek to coordinate their production decision through the Grim-Trigger Strategy considered in Example 14.5. What discount factor supports continued collusion?

- We found Cournot profit and the cartel profit for each firm from parts (a) and (b) of the exercise:

$$\pi^{Cournot} = \$190.76 \text{ and } \pi^{Cartel} = \$215.63$$

so we only need to find the profit from deviation. We do this by plugging the rival gas station's output into the gas station's best response function to find their output:

$$q(47.5) = 95 - \frac{1}{2}47.5 = 71.25 \text{ units.}$$

We can plug the two quantities into the gas station's profit to find their profit from deviating

$$\pi^{Dev.} = [10 - 0.05(71.25 + 47.5)]\, 71.25 - [10 + 0.5(71.25)] = \$243.83$$

- After a history of cooperation, the discounted stream of profits from cooperating in the cartel are

$$\begin{aligned} & 215.63 + \delta 215.63 + \delta^2 215.63 + ... \\ = \ & 215.63(1 + \delta + \delta^2 + ...) \\ = \ & 215.63\frac{1}{1-\delta} \end{aligned}$$

The discounted stream of payoffs from deviating is

$$\begin{aligned}
\underbrace{243.83}_{\text{Deviation}} + \underbrace{\delta 190.76 + \delta^2 190.76 + ...}_{\text{Punishment}} &= 243.93 + 190.76\delta(1 + \delta + \delta^2 + ...) \\
&= 243.83 + 190.76\delta\frac{1}{1-\delta} \\
&= 243.83 + 190.76\frac{\delta}{1-\delta}
\end{aligned}$$

- *Comparing profits.* Every firm i prefers to cooperate as long as

$$\frac{215.63}{1-\delta} > 243.83 + 190.76\frac{\delta}{1-\delta}$$

Multiplying each side by $(1-\delta)$, we obtain

$$215.63 > 243.83(1-\delta) + \delta 190.76$$

rearranging, $53.07\delta > 28.2$, and solving for δ, we obtain

$$\delta > 0.53$$

A discount factor larger than 0.53 will sustain a cooperation between the gas stations.