# EconS 301- Intermediate Microeconomic Theory
# Homework #4 - Due date: Tuesday October 25th, 2022.

1. Consider a firm with the Cobb-Douglas production function $f(K,L) = 4K^{1/2}L^{1/3}$, where $K$ denotes units of capital and $L$ represents units of labor. Assume that the firm faces input prices of $r = \$10$ per unit of capital, and $w = \$7$ per unit of labor.

   (a) Solve the firm's cost-minimization problem, to obtain the combination of inputs (labor and capital) that minimizes the firm's cost of producing a given amount of output, $q$.

   - The tangency condition $\frac{MP_L}{MP_K} = \frac{w}{r}$ is

$$\frac{\frac{4}{3}K^{1/2}L^{-2/3}}{2K^{-1/2}L^{1/3}} = \frac{7}{10}$$

which simplifies to

$$\frac{2}{3}\frac{K}{L} = \frac{7}{10}$$

This contains both $L$ and $K$, so we solve for $K$:

$$K = \frac{7}{10}\frac{3}{2}L = \frac{21}{20}L$$

We can insert this back into the firm's output target $q = 4K^{1/2}L^{1/3}$, to find that

$$q = 4\underbrace{\left(\frac{21}{20}L\right)}_{K}^{1/2} L^{1/3}$$

rearranging,

$$q = 4\left(\frac{21}{20}\right)^{1/2} L^{2/6+3/6},$$

where we need to solve for $L$:

$$\begin{aligned}
L^{5/6} &= \frac{q}{4}\left(\frac{20}{21}\right)^{1/2} \\
L &= \left(\frac{q}{4}\left(\frac{20}{21}\right)^{1/2}\right)^{6/5} \\
L &= \left(\frac{q}{4}\right)^{6/5}\left(\frac{20}{21}\right)^{3/5} \simeq 0.184q^{6/5}.
\end{aligned}$$

This is the firm's labor demand. Plugging this back into the tangency condition,

$$K = \frac{21}{20}\left(\frac{q}{4}\right)^{6/5}\left(\frac{20}{21}\right)^{3/5}$$

which simplifies to

$$K = \left(\frac{q}{4}\right)^{6/5}\left(\frac{21}{20}\right)^{2/5} \simeq 0.193q^{6/5}$$

which is the firm's demand for capital.

(b) Use your results from part (a) to find the firm's cost function. This is its long-run total cost, as all inputs can be altered.

- Plugging back into the cost function, we get

$$C = wL + rK = 7\underbrace{\left(\frac{q}{4}\right)^{6/5}\left(\frac{20}{21}\right)^{3/5}}_{L} + 10\underbrace{\left(\frac{q}{4}\right)^{6/5}\left(\frac{21}{20}\right)^{2/5}}_{K}.$$

If we simplify this numerically, we get

$$C \simeq 3.218q^{6/5}.$$

(c) Find the firm's marginal cost function, and its average cost function. Interpret.

- The firm's marginal cost is

$$MC = \frac{dC}{dq} = \frac{6}{5}3.218q^{1/5} \simeq 3.862q^{1/5}.$$

Each additional unit the firm produces will cost about $3.862q^{1/5}$, which increases as $q$ increases.
The firm's average cost is

$$AC = \frac{C}{q} = \frac{3.218q^{6/5}}{q} = 3.218q^{1/5}.$$

This average cost lies below the marginal cost, but has the same shape. Figure 8.1 depicts marginal and average costs.

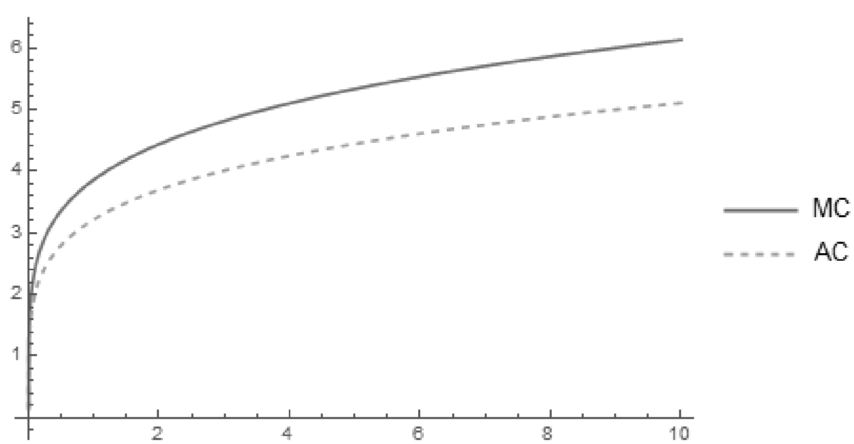


Figure 8.1. Marginal and average cost curves.

(d) Assume now that the amount of capital is held fixed at $\overline{K} = 3$ units. Solve the firm's cost-minimization problem again to find the amount of labor that minimizes the firm's cost.

- If $\overline{K} = 3$, the firm's cost-minimizing amount of labor satisfies

$$q = 4\underbrace{(3)}_{\overline{K}}{}^{1/2}L^{1/3}.$$

Solving for $L$, we find that

$$\begin{aligned} L^{1/3} &= \frac{q}{4(3)^{1/2}} \\ L &= \left(\frac{q}{4(3)^{1/2}}\right)^3 \\ L &= \frac{q^3}{64(3)^{3/2}} \simeq 0.003q^3. \end{aligned}$$

(e) Use your results from part (c) to find the firm's short-run cost function (since in the short run the firm can only alter the amount of labor, but without changing the units of capital).

- Plugging the previous result into the firm's total cost, we get

$$C = 7\underbrace{(0.003q^3)}_{L} + 10\underbrace{(3)}_{\overline{K}} = 0.021q^3 + 30.$$

2. Repeat the analysis in the previous exercise, but assuming now that the firm faces a production function $f(K, L) = 4K + L$, thus treating capital and labor as substitutes in the production process.

- The tangency condition $\frac{MP_L}{MP_K} = \frac{w}{r}$ is

$$\frac{1}{4} = \frac{7}{10}.$$

Since it includes no inputs (and is not equal), we need to compare $\frac{MP_L}{w} = \frac{1}{7}$ against $\frac{MP_K}{r} = \frac{4}{10}$, finding the latter is bigger. This means the firm only uses units of capital in its production process. We should set $L = 0$ and solve for $K$:

$$q = 4K + 0,$$

or, more simply $K = \frac{q}{4}$. This is the firm's input demand for $K$. To get the long run cost function, we plug this into the firm's costs:

$$C = 7\underbrace{(0)}_{L} + 10\underbrace{\frac{q}{4}}_{K} = \frac{5}{2}q.$$

In the short run, if $\overline{K} = 3$, then the firm's output is

$$q = 4(3) + L,$$

and solving for $L$, we get

$$L = \begin{cases} q - 12 & \text{if } q > 12 \\ 0 & \text{otherwise.} \end{cases}$$

Plugging this into the firm's cost function, we find

$$C = 7\underbrace{(q-12)}_{L} + 10\underbrace{(3)}_{\overline{K}} = 7q - 54,$$

for $q > 12$ and $C = 30$ otherwise since $C = 10(3) = 30$.

3. A publisher for textbooks has a total cost of $TC(q) = 25,000 - 50q + 15q^2$.

   (a) Find the publisher's marginal cost, average cost, average variable cost, and average fixed cost.

   - The publisher's marginal cost, average cost, average variable cost, and average fixed cost are

$$\begin{aligned} MC(q) &= \frac{\partial TC(q)}{\partial q} = 30q - 50, \\ AC(q) &= \frac{TC(q)}{q} = \frac{25,000}{q} - 50 + 15q, \\ AVC(q) &= 15q - 50, \\ AFC &= \frac{25,000}{q} \end{aligned}$$

   (b) Find the value of $q$ for where the marginal cost curve crosses the average cost curve and average variable cost curve.

   - The marginal and average cost curves cross at

$$30q - 50 = \frac{25,000}{q} - 50 + 15q,$$

   which simplifies to

$$\begin{aligned} 15q &= \frac{25,000}{q}, \\ q^2 &= \frac{25,000}{15}, \\ q &= \sqrt{\frac{25,000}{15}} \simeq 40.82. \end{aligned}$$

   The marginal and average variable cost cross where $MC = AVC$, that is

$$30q - 50 = 15q - 50$$

   which simplifies to $15q = 0$. This means that the two curves cross at $q = 0$.

(c) Find the output elasticity $\varepsilon_{TC,q}$.

- The output elasticity is

$$\begin{aligned}\varepsilon_{TC,q} &= \frac{\partial TC}{\partial q}\frac{q}{TC}, \\ &= (30q - 50)\frac{q}{25,000 - 50q + 15q^2}, \\ &= \frac{30q^2 - 50q}{25,000 - 50q + 15q^2}.\end{aligned}$$

If we evaluate the output elasticity at $q = 100$, we get $\varepsilon_{TC,q} \simeq 1.74$, which means that if output increases by 1% then total costs increase by 1.74%.

4. Consider a monopolist facing an inverse demand $p(q) = 10 - 4q$. Use the Lerner index to find the monopolist's profit-maximizing price.

- After solving for $q$ in inverse demand function $p(q) = 10 - 4q$, we obtain direct demand function $q(p) = \frac{5}{2} - \frac{1}{4}p$. The elasticity is then

$$\varepsilon_{q,p} = \frac{\partial q(p)}{\partial p}\frac{p^M}{q^M} = -\frac{1}{4}\frac{p}{\frac{5}{2} - \frac{1}{4}p} = -\frac{p}{10 - p},$$

since $\dfrac{\partial q(p)}{\partial p} = -\frac{1}{4}$. In this setting, marginal costs are $MC(q) = 4$ (see example 10.5 for details). The Lerner index, $\frac{p - MC(q)}{p} = -\frac{1}{\varepsilon_{q,p}}$, becomes

$$\frac{p - 4}{p} = -\frac{1}{-\frac{p}{10 - p}},$$

or, after rearranging, we obtain

$$\frac{p - 4}{p} = -\frac{10 - p}{p},$$

which simplifies to $p - 4 = 10 - p$ or, after solving for price $p$, $p = \$7$.

5. Consider a drug company holding the patent of a new drug for a rare disease (monopoly rights). The firm faces inverse demand function $p(q) = 100 - 0.1q$, and a cost function $C(q) = 4q$.

(a) Find the monopolist profit-maximizing output, its price, and its profits.

- *Monopoly output.* The monopolist maximizes its output when $MR(q) = MC(q)$, where marginal revenue is

$$MR(q) = p(q) + \frac{\partial p(q)}{\partial q}q = \underbrace{100 - 0.1q}_{p(q)} - (0.1)q = 100 - 0.2q,$$

since $\frac{\partial p(q)}{\partial q} = -0.1$. The marginal cost is

$$MC(q) = \frac{\partial C(q)}{\partial q} = 4.$$

Setting marginal revenue equal to marginal cost, $MR(q) = MC(q)$, we have

$$100 - 0.2q = 4.$$

Rearranging, we find $96 = 0.2q$. Solving for $q$, we obtain that the drug company produces

$$q^M = 480 \text{ units.}$$

Figure 10.4 depicts the above inverse demand curve, $p(q)$, as well as the $MR(q)$ and $MC(q)$ curves.

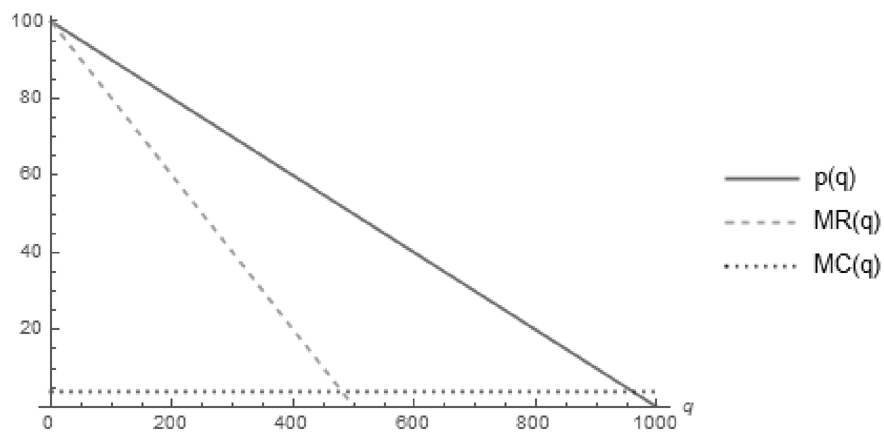


Figure 10.4. Marginal revenue and marginal cost curves with linear costs.

- *Monopoly price.* We can insert output $q^M = 480$ into the inverse demand curve to find the monopoly price

$$p^M = 100 - 0.1(480) = \$52.$$

- *Monopoly profit.* The monopolist's profit is

$$\begin{aligned} \pi^M &= p^M q^M - C(q^M) \\ &= (480 \times 52) - (4 \times 480) \\ &= \$21,600. \end{aligned}$$

(b) Find the competitive equilibrium output in this context.

- We find the competitive equilibrium by setting price equal to marginal cost; that is, $p(q) = MC(q)$, or

$$100 - 0.1q = 4.$$

Solving for $q$, we first simplify to $96 = 0.1q$, or $q^C = 960$ units.

- The competitive price will be equal to marginal cost, or

$$p^C = MC = \$4.$$

Relative to part (a), perfectly competitive output is larger than in monopoly (960 units rather than 480 units) and the price is lower (\$4 vs. \$52).

(c) Find the subsidy per unit of output that the government needs to offer the monopolist to induce the latter to produce the competitive equilibrium output you identified in part (b).

- A subsidy of $s$ per unit will impact the monopolist's marginal revenue, changing it to

$$MR(q) = p(q) + \frac{\partial p(q)}{\partial q} q + s = \underbrace{100 - 0.1q}_{p(q)} - (0.1)q + s = 100 - 0.2q + s,$$

In the monopoly equilibrium, we know that $MR(q) = MC(q)$, or

$$100 - 0.2q + s = 4.$$

The regulator wants to induce the monopolist to produce $q^C = 960$ units, so we plug this in for $q$:

$$100 - 0.2(960) + s = 4.$$

Simplifying,

$$-92 + s = 4,$$

that is, a subsidy of $s = \$96$ per unit of output.

(d) What is the total cost that the government incurs with the subsidy? How are profits affected by the subsidy (i.e., the change in profits from parts a to c)?

- The total subsidy can be calculated by multiplying the subsidy per unit of output times the quantity sold,

$$TS = sq^C = 96(960) = \$92{,}160.$$

The monopolist's profit is calculated as

$$\begin{aligned} \pi &= p^C q^C - C(q^C) + TS \\ &= \underbrace{4(960) - 4(960)}_{\text{zero}} + 92{,}160 \\ &= \$92,160. \end{aligned}$$

The monopolist's profit if it were induced to produce the competitive equilibrium will be exactly that of the subsidy. Remember that when a firm is in a competitive equilibrium, its economic profit is zero. Therefore, the regulator is inducing the monopolist to produce at a quantity where it will make no profit, but it will have to subsidize the monopolist in order to do so.