# EconS 301- Intermediate Microeconomic Theory
# Homework #3 - Due date: Thursday October 13th, 2022.

1. Consider a firm with production function $q = 5L + 2K$. If the firm seeks to produce $q = 230$ units, find the isoquant, its vertical and horizontal intercept, and its slope.

   - Solving for $K$ in our production function $230 = 5L + 2K$ yields the isoquant

   $$K = \frac{230}{2} - \frac{5}{2}L.$$

   Its vertical axis will be where $K = \dfrac{230}{2} = 115$, and its horizontal axis will be at the level of $L$ that solves $0 = \dfrac{230}{2} - \dfrac{5}{2}L$; that is, $L = \dfrac{230}{5} = 46$ workers. The slope is constant and is equal to the $MRTS$, $\dfrac{5}{2}$.

2. Consider a firm with production function $q = 5L^{1/3}K^{2/3}$. Find the firm's MRTS. Then, assuming that the firm seeks to produce $q = 220$ units of output, find and depict the isoquant.

   - The $MRTS$ is

   $$MRTS = \frac{MP_L}{MP_K} = \frac{\frac{5}{3}L^{-2/3}K^{2/3}}{\frac{10}{3}L^{1/3}K^{-1/3}} = \frac{K^{\frac{2}{3}-\left(\frac{1}{3}\right)}}{2L^{\frac{1}{3}-\left(\frac{2}{3}\right)}} = \frac{K}{2L}.$$

   - To find the isoquant, we need to solve $200 = 5L^{1/3}K^{2/3}$ for $K$. Dividing each side by 5 gives us $40 = L^{1/3}K^{2/3}$. Cubing both sides yields $40^3 = LK^2$. Next, divide by $L$ to get $K^2 = \dfrac{40^3}{L}$, and square-root each side to get the isoquant

   $$K = \frac{40^{3/2}}{L^{1/2}}.$$

   This isoquant will approach both axes, but never cross them; however, it will approach the $K$-axis more quickly than the $L$-axis.

3. Sarah is looking into producing her homemade dog treats on a larger scale and is contemplating two different kitchen sizes ($K$). Her production of dog treats follows $q = 200KL + K^2L^3$. What are the marginal and average product curves for labor when $K = 5$? What happens to the marginal and average product curves when her kitchen doubles to $K = 10$?

   - The marginal product curve is found by differentiating her production function with respect to $L$:

   $$MP_L = \frac{\partial q}{\partial L} = 200K + 3K^2L^2.$$

The average product curve is found by dividing the production function by labor:

$$AP_L = \frac{q}{L} = \frac{200KL}{L} + \frac{K^2L^3}{L} = 200K + K^2L^2.$$

At $K = 5$ units of capital,

$$MP_L = 200(5) + 3(5)^2L^2 = 1000 + 75L^2, \text{ and}$$

$$AP_L = 200(5) + (5)^2L^2 = 1000 + 25L^2.$$

Figure.1 depicts these two curves.

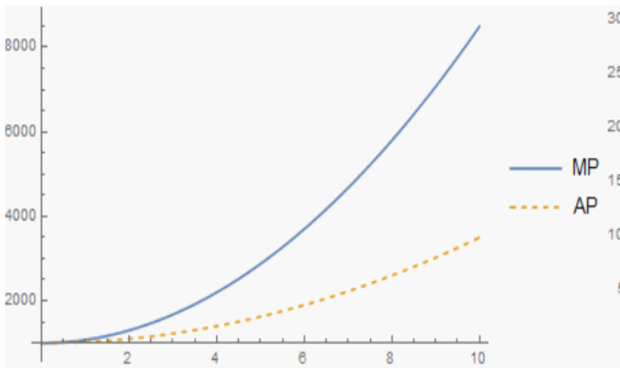

Figure 1. $MP$ and $AP$ curves at $K = 5$.

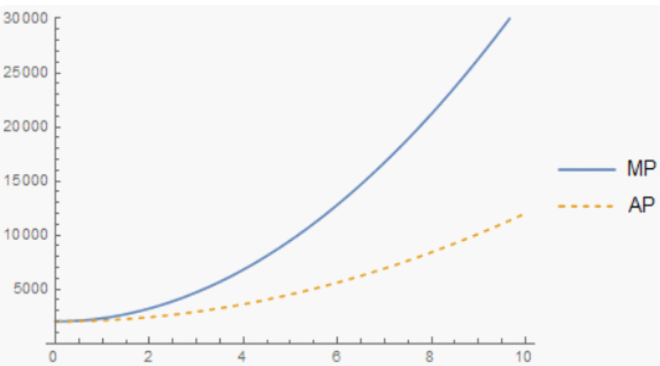

Figure 2. $MP$ and $AP$ curves at $K = 10$.

At $K = 10$ units of capital,

$$MP_L = 200(10) + 3(10)^2L^2 = 2000 + 300L^2, \text{ and}$$

$$AP_L = 200(10) + (10)^2L^2 = 2000 + 100L^2.$$

Figure 2 depicts these two curves.

4. Is it possible for two firms with different Cobb-Douglas production functions, $q_1 = AL^\alpha K^\beta$ and $q_2 = L^\alpha K^\beta$, to have the same marginal products at particular levels of capital and labor? What about each firms' marginal rate of technical substitution.

   - The marginal product of labor for the first production function, $q_1$, is

     $$MP_{L1} = \frac{\partial q_1}{\partial L} = \alpha AL^{\alpha-1}K^\beta,$$

     and for the second production function, $q_2$, is

     $$MP_{L2} = \frac{\partial q_1}{\partial L} = \alpha L^{\alpha-1}K^\beta$$

     Comparing we see that $MP_{L1} = MP_{L2}$ if

     $$\alpha AL^\alpha K^\beta = \alpha L^\alpha K^\beta,$$

which happens when $A = 1$. Therefore, the only time the marginal products will be equal is when they are the same production function. A similar argument applies for the marginal product of capital.

However, the MRTS between the two will always coincide. To see this,

$$\begin{aligned} MRTS_{q1} = \frac{\alpha A L^{\alpha-1} K^{\beta}}{\beta A L^{\alpha} K^{\beta-1}} &= \frac{\alpha L^{\alpha-1} K^{\beta}}{\beta L^{\alpha} K^{\beta-1}} = MRTS_{q2} \\ \frac{\alpha L^{\alpha-1} K^{\beta}}{\beta L^{\alpha} K^{\beta-1}} &= \frac{\alpha L^{\alpha-1} K^{\beta}}{\beta L^{\alpha} K^{\beta-1}} \\ \frac{\alpha K}{\beta L} &= \frac{\alpha K}{\beta L}, \end{aligned}$$

which coincides for all levels of capital and labor.

5. A firm has the following cost function:

$$TC(q) = 2q^3 - \frac{1}{3}q^2 + \frac{1}{2}q + \frac{9}{10},$$

where $q$ denotes units of output. Intuitively, the first three terms on the right side capture the firm's variable cost, because they depend on the output the firm produces, whereas the last term represents its fixed cost, as it is not a function of output $q$.

(a) *Total cost.* For which output $q$ does the total cost curve $TC(q)$ increase or decrease? For which values is it concave or convex in output?

- We need to look at the derivative:

$$\frac{dC}{dq} = 6q^2 - \frac{2}{3}q + \frac{1}{2}.$$

This is always greater than 0, which means that the total cost curve is always increasing. To evaluate the concavity, we look at the second derivative:

$$\frac{d^2C}{dq^2} = 12q - \frac{2}{3}.$$

Setting equal to zero and solving for $q$, we can find that the second derivative is positive for $q > \frac{1}{18}$, which is where the total cost curve is convex. For $q < \frac{1}{18}$, the total cost curve is concave.

(b) *Marginal cost.* For which output $q$ does the marginal cost curve $\frac{\partial TC(q)}{\partial q}$ increase or decrease? For which values is it concave or convex in output?

- We found marginal cost in the last problem, $MC = 6q^2 - \frac{2}{3}q + \frac{1}{2}$, and its derivative, $12q - \frac{2}{3}$, which tells us that marginal cost is increasing for $q > \frac{1}{18}$. The second derivative of marginal cost, $\frac{d^2MC}{dq^2} = 12 > 0$ means that the marginal cost curve is convex.

(c) *Average cost.* For which output $q$ does the average cost curve $AC(q) = \frac{TC(q)}{q}$ increase or decrease? For which values is it concave or convex in output?

- Average cost is
$$\frac{C(q)}{q} = 2q^2 - \frac{1}{3}q + \frac{1}{2} + \frac{9}{10q}.$$
The derivative of average cost is
$$\frac{dAC}{dq} = 4q - \frac{1}{3} - \frac{9}{10q^2}.$$
Solving for when this equals zero, we find that the average cost is decreasing for $q < 0.52$ but increasing for $q > 0.52$. The second derivative tells us about its concavity:
$$\frac{d^2AC}{dq^2} = 4 + \frac{18}{10q},$$
which is always positive, so average cost is convex.

(d) *Average variable cost.* For which output $q$ does the average variable cost curve $AC(q)$ increase or decrease? For which values is it concave or convex in output?

- The variable cost is part of costs that varies with output: $2q^3 - \frac{1}{3}q^2 + \frac{1}{2}q$. Therefore, average variable cost is:
$$AVC = 2q^2 - \frac{1}{3}q + \frac{1}{2}.$$
The derivative will tell us when it increases or decreases:
$$\frac{dAVC}{dq} = 4q - \frac{1}{3}.$$
Setting equal to zero and solving, we find that average variable cost decreases when $q < \frac{1}{12} \simeq 0.083$, and increases at larger values of $q$. The second derivative, $\frac{d^2AVC}{dq^2} = 4$, tells us that the average variable cost is convex.

(e) Find the value of $q$ where the marginal cost curve crosses the total cost curve, where it crosses the average cost curve, and where it crosses the average variable cost curve.

- The marginal cost curve crosses the average cost and average variable cost curves at their lowest points, which we have already calculated. Marginal and average costs cross at $q = 0.52$, while marginal and average variable costs cross at $q = 0.083$.