EconS 301- Intermediate Microeconomic Theory
Recitation - Friday September 23rd, 2022.

1. Consider that Eric's utility function is $u(x,y) = 2x + 3y$, which is just an example of $u(x,y) = ax + by$, where $a = 2$ and $b = 3$. Show that this utility function satisfies completeness, transitivity, monotonicity, strict monotonicity, and nonsatiation. Then consider one of Eric's friends, John, who has a utility function $u(x,y) = \min\{2x, 3y\}$. Show that the utility function satisfies all the properties, except for strict monotonicity.

   - Starting with Eric's utility function:
     - *Completeness.* Like in example 2.4 in the textbook, as long as we can specify two bundles $A = (x_A, y_A)$ and $B = (x_B, y_B)$ and rank them such that either $u(x_A, y_A) \geq u(x_B, y_B)$, $u(x_B, y_B) \geq u(x_A, y_A)$, or both, completeness is satisfied. For any values of x and y, we obtain a real value for the utility function $u(x,y) = 2x + 3y$, so we can compare any pair of bundles by simply comparing the utility values. Thus, $u(x,y)$ satisfies completeness.
     - *Transitivity.* Again, for every three bundles A, B and C, where $u(x_A, y_A) \geq u(x_B, y_B)$ and $u(x_B, y_B) \geq u(x_C, y_C)$, we must have that $u(x_A, y_A) \geq u(x_C, y_C)$ for transitivity to hold. Just like we did for completeness, we can translate any pair of values for x and y into a corresponding utility value, then rank them accordingly. Thus, transitivity holds by definition.
     - *Strict Monotonicity.* An increase in either the value of x or y for Eric's utility function leads to a strictly higher value of his utility. Thus, strict monotonicity holds.
     - *Monotonicity.* Since Eric's utility function is strictly monotonic, it is also monotonic. Intuitively, if we increase both the values of x and y in Eric's utility function, Eric receives a strictly higher utility level.
     - *Nonsatiation.* It is satisfied by monotonicity. If we increase Eric's bundle from (x, y) to $(x + b, y + a)$ for any positive values of a and b (i.e., increasing the amount of good x by b units, and the amount of good y by a units), Eric receives a strictly higher utility level. Thus, Eric never reaches a bliss point and his preferences satisfy nonsatiation.
   - For John's utility function, we follow a similar process.
     - *Completeness.* For completeness to hold, we need that, for any two bundles $A = (x_A, y_A)$ and $B = (x_B, y_B)$, we can rank them such that either $u(x_A, y_A) \geq u(x_B, y_B)$, $u(x_B, y_B) \geq u(x_A, y_A)$, or both. For any real values of x and y, we obtain a real value for the utility function $u(x,y) = \min\{2x, 3y\}$, and can compare any pair of bundles by simply comparing their utility values. Thus, $u(x,y)$ satisfies completeness.
     - *Transitivity.* Since we can translate any possible bundle into a numerical utility value, we can rank them appropriately. If bundle A is preferred to bundle B and bundle B is preferred to bundle C, then $u(x_A, y_A) > u(x_B, y_B)$ and $u(x_B, y_B) > u(x_C, y_C)$. This implies that $u(x_A, y_A) > u(x_C, y_C)$ and thus bundle A must be preferred to bundle C and $u(x,y)$ satisfies transitivity.

- *Strict Monotonicity.* John's utility function does not satisfy strict monotonicity. Suppose $x = 1$ and $y = 10$. John's utility level is

$$u(1, 10) = \min\{2(1), 3(10)\} = \min\{2, 30\} = 2.$$

  If we increase the amount of good y to 11 units, we obtain the same utility value:

$$u(1, 11) = \min\{2(1), 3(11)\} = \min\{2, 33\} = 2.$$

  Intuitively, John prefers to consume goods x and y in fixed proportions, and increasing an overly proportioned good (good y in this case) does not increase John's utility. Since an increase of one component of John's bundle does not necessarily lead to a more preferred bundle, John's utility function does not satisfy strict monotonicity.

- *Monotonicity.* If we increase both the values of x and y in John's utility function, the lesser between $2x$ and $3y$ must increase as well. To see this point with our above example, consider bundle $(1, 10)$, which yields a utility level of

$$u(1, 10) = \min\{2(1), 3(10)\} = \min\{2, 30\} = 2,$$

  and bundle $(2, 11)$, where both goods increased by one unit, which yields a utility level

$$u(2, 11) = \min\{2(2), 3(11)\} = \min\{4, 33\} = 4.$$

  Thus, John reaches a strictly higher utility level and his utility function satisfies monotonicity.

- *Nonsatiation.* Since John's utility function satisfies monotonicity, it satisfies nonsatiation. If we increase John's bundle from (x, y) to $(x + b, y + a)$ for any positive values of a and b (i.e., increasing the amount of good x by b units, and the amount of good y by a units), the lesser of $2x$ and $3y$ must also increase and John receives a strictly higher utility level. Thus, John never reaches a bliss point and his preferences satisfy nonsatiation.

2. Ana's utility function is $u(x, y) = 5\,(x-2)^{1/2}\,(y-1)^{1/3}$. Find her marginal utilities and her MRS, and check if it is decreasing in x.

- Starting with Ana's marginal utilities, we must use the chain rule.

  - The marginal utility of good x is

$$MU_x = \frac{\partial u(x, y)}{\partial x} = \frac{5}{2}(x-2)^{-1/2}(1)(y-1)^{1/3}.$$

  - Likewise, the marginal utility of good y is

$$MU_y = \frac{\partial u(x, y)}{\partial y} = \frac{5}{3}(x-2)^{1/2}(y-1)^{-2/3}(1).$$

- Putting our two marginal utilities together, we can obtain her marginal rate of substitution,

$$MRS_{x,y} = -\frac{MU_x}{MU_y} = -\frac{\frac{5}{2}(x-2)^{-1/2}(y-1)^{1/3}}{\frac{5}{3}(x-2)^{1/2}(y-1)^{-2/3}} = -\frac{2}{3}\frac{y-1}{x-2}.$$

  - Since good x (good y) appears solely in the denominator (numerator) of the marginal rate of substitution, the $MRS_{x,y}$ is decreasing (increasing) in good x (good y, respectively).

3. Consider an individual with the Cobb-Douglas utility function

$$u(x,y) = \sqrt{x}\sqrt{y}.$$

Assume that her income is $I = \$120$, the price of good x is $p_x = \$4$, and the price of good y is $p_y = \$10$.

(a) Find the marginal utility of good x, MU_x, and that of good y, MU_y.

- We calculate the marginal utilities by differentiating with respect to each respective variable,

$$MU_x = \frac{\partial u(x,y)}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} \quad \text{and} \quad MU_y = \frac{\partial u(x,y)}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}}.$$

(b) Given the results in part (a), does this utility function satisfy monotonicity? What about strict monotonicity?

- Since both marginal utilities are strictly positive when the individual consumes a positive amount of both goods, this utility function satisfies both monotonicity and strict monotonicity. If the individual consumes zero units of one (or both) goods, the utility function satisfies monotonicity, but violates strict monotonicity.

(c) Using the marginal utilities you found in part (a), find the MRS.

- The marginal rate of substitution can be found by taking the ratio of our marginal utility with respect to x to our marginal utility with respect to y,

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{\frac{\sqrt{y}}{2\sqrt{x}}}{\frac{\sqrt{x}}{2\sqrt{y}}} = \frac{\sqrt{y}\,(2\sqrt{y})}{\sqrt{x}\,(2\sqrt{x})} = \frac{2y^{\frac{1}{2}+\frac{1}{2}}}{2x^{\frac{1}{2}+\frac{1}{2}}} = \frac{y}{x}.$$

4. Consider again Eric's demand function, $x = \frac{5I}{\sqrt{p_x - 3p_y}}$. Find its price elasticity ε_{x,p_x}, and interpret your result.

- Eric's price elasticity can be calculated using the standard formula,

$$\varepsilon_{x,p_x} = \frac{\partial x(p_x,p_y,I)}{\partial p_x}\frac{p_x}{x(p_x,p_y,I)},$$

and using our result for the derivative of the demand for good x with respect to its price from self-assessment 4.5, we have

$$\varepsilon_{x,p_x} = -\frac{5I}{2\sqrt{p_x}\left(\sqrt{p_x} - 3p_y\right)^2} \frac{p_x}{\frac{5I}{\sqrt{p_x}-3p_y}},$$

which we can rearrange to obtain

$$\varepsilon_{x,p_x} = -\frac{5Ip_x\left(\sqrt{p_x} - 3p_y\right)}{10I\sqrt{p_x}\left(\sqrt{p_x} - 3p_y\right)^2}.$$

We can simplify this expression, dividing both the numerator and denominator by $5I\sqrt{p_x}\left(\sqrt{p_x} - 3p_y\right)$ to obtain

$$\varepsilon_{x,p_x} = -\frac{\sqrt{p_x}}{2\left(\sqrt{p_x} - 3p_y\right)},$$

which implies that as the price of good x increases by 1 percent, the quantity demanded of good x decreases by $\frac{\sqrt{p_x}}{2\left(\sqrt{p_x}-3p_y\right)}$ percent. This expression is negative if the term in the denominator is positive; that is, $\sqrt{p_x} > 3p_y$ has to be positive for demand for good x to be positive, or $\frac{\sqrt{p_x}}{3} > p_y$. We next present two numerical examples:

- *Good x is relatively expensive.* If $p_x = \$9$ and $p_y = \$\frac{1}{2}$ (which satisfies condition $\frac{\sqrt{p_x}}{3} > p_y$ since $\frac{\sqrt{9}}{3} = 1 > \frac{1}{2} = p_y$), this elasticity becomes

$$\varepsilon_{x,p_x} = -\frac{\sqrt{9}}{2\left(\sqrt{9} - (3 \times \frac{1}{2})\right)} = -\frac{3}{2\,(3 - 1.5)} = -1,$$

  implying that a 1 percent increase in p_x increases the quantity demanded of good x proportionally (i.e., by 1 percent).
- *Good x is relatively inexpensive.* If, instead, prices are $p_x = \$2$ and $p_y = \$1$, condition $\frac{\sqrt{p_x}}{3} > p_y$ is violated since $\frac{\sqrt{2}}{3} = 0.47 < 1 = p_y$), this elasticity becomes

$$\varepsilon_{x,p_x} = -\frac{\sqrt{2}}{2\left(\sqrt{2} - (3 \times 1)\right)} = \frac{3}{2\left(3 - \sqrt{2}\right)} \simeq 0.45,$$

  meaning that, after a 1 percent increase in p_x, the consumer increases her quantity demanded for this product.

Overall, after a 1 percent increase in p_x, the consumer responds by reducing her quantity demanded when good x is relatively expensive (that is, when condition $\frac{\sqrt{p_x}}{3} > p_y$ holds; but increases her quantity demanded when good x is relatively inexpensive.

5. Chelsea's utility function is $u(x, y) = 3x + 4y^{1/2}$, her income is $I = \$220$, and $p_y = \$1$. The price of good x decreases from $p_x = \$3$ to $p_x' = \$2$. Find the substitution and income effects.

- To begin our analysis, it is useful to consider Chelsea's marginal rate of substitution,

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{3}{2y^{-1/2}} = \frac{3\sqrt{y}}{2}.$$

- *Finding initial bundle A.* Using Chelsea's initial prices, we can establish her tangency condition, $\frac{MU_x}{MU_y} = \frac{3}{1}$, as

$$\frac{3\sqrt{y}}{2} = 3.$$

Solving this expression for x, we obtain

$$y = (2)^2 = 4 \text{ units}.$$

Using Chelsea's budget line, we can substitute this result in to obtain

$$3x + 4 = 220.$$

Solving this expression for x, we obtain Chelsea's initial consumption of good y,

$$3x = 220 - 4 \quad x = 72 \text{ units}.$$

This gives us Chelsea's initial bundle $A = (72, 4)$.

- *Finding final bundle C.* Now with Chelsea's final prices, her tangency condition, $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$ is

$$\frac{3\sqrt{y}}{2} = \frac{2}{1}.$$

Once again, we can solve this expression for x to obtain

$$y = \left(\frac{4}{3}\right)^2 = \frac{16}{9} = 1.78 \text{ units}.$$

As before, we use Chelsea's budget line and substitute this result in to obtain

$$2x + 1.78 = 220.$$

Solving this expression for y, we obtain Chelsea's final consumption of good y,

$$x = \frac{218.22}{2} = 109.11 \text{ units},$$

which provides Chelsea's final bundle $C = (109.11, 1.77)$.

- *Finding decomposition bundle B.* Now we need to determine how Chelsea's bundle changes from her initial bundle to her final bundle, breaking them into the substitution and income effects. To calculate this, we need to satisfy both conditions as explained in example 4.9.

- First, the decomposition bundle must reach the utility level Chelsea received under her initial bundle, which we can calculate by inserting bundle A into Chelsea's utility function,

  $$3 \times 72 + 4(\sqrt{4}) = 224$$

  Thus, Chelsea's decomposition bundle B must satisfy $3x + 4y^{1/2} = 224$.
- Second, the decomposition bundle must be tangent to Chelsea's indifference curve. This happens where the slope of Chelsea's indifference curve is equal to the slope of her final budget line, which implies that we use the tangency condition from Chelsea's final bundle, $y = 1.77$.

Substituting Chelsea's tangency condition into the first condition gives

$$3x + 4(\sqrt{1.77}) = 224.$$

Since $4\,(1.77)^{1/2} = 5.34$, we can solve this expression for good y to obtain,

$$x = \frac{218.66}{3} = 72.88 \text{ units.}$$

This gives us Chelsea's decomposition bundle $B = (72.88, 1.77)$.

- Last, we can calculate Chelsea's substitution and income effects by comparing the values of her different bundles:

$$\begin{aligned} \text{Substitution Effect:} &\quad x_B - x_A = 72.88 - 72 = 0.88, \text{ and} \\ \text{Income Effect:} &\quad x_C - x_B = 109.11 - 72.88 = 36.23. \end{aligned}$$

This result for Chelsea should make sense. Since the majority of her utility comes from consuming good x (the linear parameter), Chelsea allocates all of her savings from the price change to purchasing more of good x. This does not lead to zero income effect.