Homework # 8 - [Due on December 1st, 2021]

1. Consider a setting with N individuals, each of them simultaneously and independently deciding how many dollars to contribute to a public good. Assume that each individual has a Cobb-Douglas utility function $u(x_i, G) = x_i^{1-\alpha} G^{\alpha}$ where $G = \sum_{j=1}^{n} g_j$ denotes aggregate contributions and $\alpha \in (0, 1)$ for all $i = 1, ..., N$. For simplicity, normalize the price of the public good.

   (a) Set up the utility maximization problem of agent i. Find the demand functions denoted $(x_i(\cdot), G(\cdot))$, for the private and public good.

   - Agent i's maximization problem is

$$\max_{x_i, g_i} \quad x_i^{1-\alpha} G^{\alpha} = x_i^{1-\alpha} \underbrace{\left(g_i + \sum_{i \neq j} g_j \right)^{\alpha}}_{G}$$

$$\text{subject to} \quad px_i + g_i = \omega_i$$

since $\frac{p_2}{p_2} = 1$ and $\frac{p_1}{p_2} \equiv p$. Inserting the constraint into the objective function, we obtain the following (unconstrained) problem

$$\max_{x_i} \quad x_i^{1-\alpha} \left(\omega_i - px_i + \sum_{i \neq j} g_j \right)^{\alpha}$$

Differentiating with respect to x_i, we find

$$(1-\alpha) x_i^{-\alpha} \left(\omega_i - px_i + \sum_{i \neq j} g_j \right)^{\alpha} - \alpha p x_i^{1-\alpha} \left(\omega_i - px_i + \sum_{i \neq j} g_j \right)^{\alpha - 1} = 0$$

which, after regarranging, yields

$$(1-\alpha) \left(\omega_i + \sum_{i \neq j} g_j \right) = px_i$$

or

$$\underbrace{\omega_i - px_i}_{g_i} = \alpha \left(\omega_i + \sum_{i \neq j} g_j \right) - \sum_{i \neq j} g_j$$

Using $px_i + g_i = \omega_i$ on the left-hand side of the above equation, or $g_i = \omega_i - px_i$, we find consumer i's demand for the public good:

$$g_i = \alpha\omega_i - (1-\alpha)\sum_{i\neq j} g_j$$

- Intuitively, this expression is consumer i's best response function, as it identifies the utility-maximizing contribution he makes to the public good as a function of the donation of all other donors, $\sum_{i\neq j} g_j$. As usual, an increase in other donors' contributions decreases individual i's donation by $(1-\alpha)$. If no other contributor donates the public good, i.e., $\sum_{i\neq j} g_j = 0$, his donation becomes a fraction of his wealth, i.e., $g_i^* = \alpha\omega_i$.

(b) Suppose that individuals are ranked according to wealth, whereby $\omega_1 \geq \omega_2 \geq ... \geq \omega_n$. Find conditions on ω_i and α for an equilibrium in which $g_2^* = ... = g_n^* = 0$ and agent 1 is the only contributor (only the richest individual contributes).

- To solve this, we only need to find conditions for which agent 2 will contribute 0 to the public good. If he does not contribute, then all other agents $j > 2$, having less wealth than agent 2, will not contribute to the public good either.
- For agent 2, she will contribute zero to the public good if

$$g_2^* = \alpha\omega_2 - (1-\alpha)g_1^* \leq 0$$

and solving this expression for g_1^* yields $g_1^* \geq \frac{\alpha}{1-\alpha}\omega_2$. Likewise, we have agent 1's contribution to the public good $g_1^* = \alpha\omega_1$. Hence, in order for agent 2 to not contribute to the public good, we must have

$$g_1^* = \alpha\omega_1 \geq \frac{\alpha}{1-\alpha}\omega_2$$

or, solving for ω_1,

$$\omega_1 \geq \frac{1}{1-\alpha}\omega_2$$

For instance, if $\alpha = \frac{1}{2}$, then $\omega_1 \geq 2\omega_2$, i.e., individual 1 must be twice as rich as individual 2 (the second richest individual in the population) for him to be the only donor.

(c) Let G_k denote aggregate donations in equilibrium when the total wealth W is divided equally among k individuals.

1. Suppose first that we divide the wealth W among 2 individuals. Find aggregate donations in this case, G_2, and show that they are lower than aggregate donations when a single individual holds all the wealth, whereby $G_2 < G_1$.

- When a single individual holds all the wealth, $\omega_1 = W$, his contribution becomes $g_1^* = \alpha W$, which coincides with aggregate donations, i.e., $G_1 = g_1^* = \alpha W$.
- For two individuals, their equilibrium bundles can be shown from part (a) as

$$\begin{aligned} x_i^* &= \frac{(1-\alpha)\left(\frac{W}{2}+g_j\right)}{p} \quad \text{for the private good, and} \\ g_i^* &= \alpha\frac{W}{2} - (1-\alpha)g_j \quad \text{for the public good, where } \; i,j = 1,2 \text{ and } i \neq j \end{aligned}$$

We can aggregate the individual contributions to the public good to find

$$G_2 = \sum_{k=1}^{2} g_k^* = \alpha W - (1-\alpha)\overbrace{(g_1+g_2)}^{G_2}$$

and solving for G_2 yields

$$G_2 = \frac{\alpha W}{2-\alpha} < \alpha W = G_1$$

as required.

2. More generally, suppose that the wealth is divided into k equal shares $\frac{W}{k}$ among k consumers. Compute the equilibrium value of G_k and show that $G_k \to 0$ when $k \to +\infty$. (The smallest amount of public production is supplied when everyone is a contributor).
   - From part (a),

$$\begin{aligned} x_i^* &= \frac{(1-\alpha)\left(\frac{W}{k}+\sum_{i\neq j} g_j\right)}{p} \quad \text{for the private good, and} \\ g_i^* &= \alpha\frac{W}{k} - (1-\alpha)\sum_{i\neq j} g_j \quad \text{for the public good, where } i,j = 1,2 \text{ and } i \neq j \end{aligned}$$

And aggregating the individual donations to the public good, we find

$$G_k = \sum_{k=1}^{k} g_k^* = \alpha W - (1-\alpha)(k-1)G_k$$

where the $(k-1)G_k$ term comes from the fact that for all k agents, $k-1$ provisions of the public good are added together, with each individual provision omitted exactly once. This creates $k-1$ complete sets of the provision. Solving for G_k yields an aggregate contribution of

$$G_k = \frac{\alpha W}{1 + (1-\alpha)(k-1)}$$

- Evaluating the limit of G_k as $k \to +\infty$, we obtain

$$\lim_{k \to +\infty} \frac{\alpha W}{1 + (1-\alpha)(k-1)} = 0$$

as required.

2. Consider a monopolist facing inverse demand function $p(q) = 1 - q$; a supply function of $q = ax$, where x denotes the number of input that the monopolist hires (e.g., labor) and $a > 0$; and cost function $C(x) = bx + dx^2$, where $b, d > 0$, thus being increasing and convex in input units x.

   (a) Write down the monopolist's profit-maximization problem. Find the equilibrium values of the monopolist's input decision, and its output level.

   - The monopolist solves

$$\max_{x \geq 0} \pi = \underbrace{p(q)q}_{\text{Revenues}} - \underbrace{\left(bx + dx^2\right)}_{\text{Costs}} = (1 - ax)ax - \left(bx + dx^2\right)$$

Differentiating with respect to x, yields

$$-b - a^2x - 2dx + a(1 - ax) = 0$$

and solving for x, we obtain an equilibrium input level of

$$x^* = \frac{a - b}{a^2 + d}$$

Since $q = ax$ by definition, the equilibrium output that the monopolist produces is

$$q^* = ax^* = \frac{a(a-b)}{a^2 + d}.$$

   (b) Assume now that the firm operates in a perfectly competitive industry, where price equals marginal cost. Find in this context the equilibrium values of the monopolist's input decision, and its output level.

   - Setting $p(q) = MC(q)$ in this perfectly competitive market, we obtain

$$1 - q = b + 2dx,$$

and since $q = ax$ by definition, we can rewrite the above equality as

$$1 - ax = b + 2dx.$$

Solving for x, we find a perfectly-competitive input demand of

$$x^C = \frac{1-b}{a+2d}$$

Therefore, equilibrium output in this context becomes

$$q^C = ax^C = \frac{a(1-b)}{a+2d}.$$

3. Consider a monopolist facing a linear inverse demand function $p(q) = a - q$ and a cost function $C(q) = (1-\alpha)q + \alpha q^2$, where $a > 2$ is the market size and $\alpha \in [0,1]$ denotes the firm's efficiency. In particular, when $\alpha = 0$, the monopolist's costs are linear, $C(q) = q$, while when $\alpha = 1$, its costs become convex, $C(q) = q^2$.

   (a) Find the monopolist's profit-maximizing output q^m and equilibrium profits π^m.

   - The monopolist chooses q to solve the following profit maximization problem,

     $$\max_{q \geq 0} \pi(q) = (a-q)q - (1-\alpha)q - \alpha q^2$$

     Differentiating with respect to q, and assuming interior solutions,

     $$a - 2q - 1 + \alpha - 2\alpha q = 0$$

     Solving for q, we obtain the monopolist's profit-maximizing output,

     $$q^m = \frac{a-1+\alpha}{2(1+\alpha)}$$

     which is positive since $a > 2$ and $\alpha \in [0,1]$ by assumption.

   - Substituting q^m into the monopolist's profit function, we obtain

     $$\begin{aligned}\pi^m &= (a - 1 + \alpha - (1+\alpha)q^m)q^m \\ &= \frac{(a-1+\alpha)^2}{4(1+\alpha)}.\end{aligned}$$

   (b) How does equilibrium output change with a and α? Explain.

- Differentiating q^m with respect to a, we have

$$\frac{\partial q^m}{\partial a} = \frac{1}{2(1+\alpha)} > 0$$

so that the monopolist produces more output as the market size increases.

- Differentiating q^m with respect to α, we find

$$\frac{\partial q^m}{\partial \alpha} = \frac{2-a}{2(1+\alpha)^2}$$

which is negative since $a > 2$ by definition. Therefore, as the cost function becomes more convex (α increases), the monopolist produces less output.

(c) How do equilibrium profits change with a and α? Explain.

- Differentiating π^m with respect to a, we have

$$\frac{\partial \pi^m}{\partial a} = \frac{a-1+\alpha}{2(1+\alpha)} > 0$$

so that the monopolist earns higher profits as the market size increases.

- Differentiating π^m with respect to α, we find

$$\frac{\partial \pi^m}{\partial \alpha} = \frac{(a-1+\alpha)(3+\alpha-a)}{4(1+\alpha)^2}$$

which is negative if $a > 3+\alpha$, so that the monopolist earns lower profits if the cost function becomes more convex (α increases) and the market is sufficiently large. Otherwise, if the market is relatively small and/or its costs are sufficiently convex, $a < 3+\alpha$, an increase in parameter α produces an increase in equilibrium profit π^m.

(d) Compare the equilibrium output found in part (a) to the perfectly competitive level q^*.

- The perfectly competitive output level q^* solves $p(q) = MC(q)$, where

$$a - q = 1 - \alpha + 2\alpha q.$$

Solving for q, the competitive output level is

$$q^* = \frac{a-1+\alpha}{1+2\alpha}.$$

- It is straightforward to verify that $q^* > q^m$ since

$$\frac{a-1+\alpha}{1+2\alpha} > \frac{a-1+\alpha}{2(1+\alpha)}$$

simplifies to $2+2\alpha > 1+2\alpha$ that holds for all values of $\alpha \in [0,1]$, indicating that the monopolist reduces output relative to the perfectly competitive level.

(e) *Numerical example.* Evaluate equilibrium output and profits in part (a) when $a = 3$ under three values of α: (i) $\alpha = 0$, (ii) $\alpha = 1/2$, and (iii) $\alpha = 1$. Compare and interpret your results.

- Substituting $a = 3$ into equilibrium output and profits in part (a), yields

$$q^m = \frac{2+\alpha}{2(1+\alpha)}$$
$$\pi^m = \frac{(2+\alpha)^2}{4(1+\alpha)}$$

- When $\alpha = 0$, $C(q) = q$ which is a linear cost function with marginal cost of 1, yielding

$$q^m = \frac{2+0}{2(1+0)} = 1$$
$$\pi^m = \frac{(2+0)^2}{4(1+0)} = 1$$

- When $\alpha = 1/2$, $C(q) = \frac{1}{2}q(1+q)$ with marginal cost of $\frac{1+2q}{2}$, so that

$$q^m = \frac{2+\frac{1}{2}}{2\left(1+\frac{1}{2}\right)} = \frac{5}{6} \approx 0.83$$
$$\pi^m = \frac{\left(2+\frac{1}{2}\right)^2}{4\left(1+\frac{1}{2}\right)} = \frac{25}{24} \approx 1.04$$

- When $\alpha = 1$, $C(q) = q^2$ with marginal cost of $2q$, resulting in

$$q^m = \frac{2+1}{2(1+1)} = \frac{3}{4} = 0.75$$
$$\pi^m = \frac{(2+1)^2}{4(1+1)} = \frac{9}{8} = 1.125$$

Comparing across the three cases where $\alpha = 0$, $\alpha = 1/2$, and $\alpha = 1$, we see that output decreases as the cost function becomes more convex. However, since $a = 3 \leq 3+\alpha$, we see that profits increase with α. This happens because the monopolist can charge a higher price on every unit of output that more than offsets the reduced units of output, ultimately allowing the monopolist to enjoy more profits even if it becomes less efficient.

4. Ann's total demand for good x is given by $x_A(p) = a - \theta_A p$, and Bob's total demand is $x_B(p) = a - \theta_B p$, where $\theta_A < \theta_B$. Intuitively, Bob's demand is more sensitive to a given increase in prices than Ann's. Alternatively, inverting these demand functions we obtain $p(q^A) = \frac{a}{\theta_A} - \frac{1}{\theta_A}x_A$ and $p(q^B) = \frac{a}{\theta_B} - \frac{1}{\theta_B}x_B$ for Ann and Bob, respectively. Hence, if $\theta_A < \theta_B$, then $\frac{a}{\theta_A} > \frac{a}{\theta_B}$, which ultimately implies that Ann's willingness to pay for the good is higher than Bob's. Finally, the (constant) marginal cost of production is $c > 0$.

(a) Suppose that the market for good x is competitive. Find the equilibrium quantity and price.

- In a competitive equilibrium we must have

$$p^B = p^A = c,$$

and therefore $x_B = a - \theta_B c$, and $x_A = a - \theta_A c$.

(b) Suppose, instead, that the firm is a monopolist. If this firm is prohibited from discriminating, what is its profit maximizing price? Under which conditions do Ann and Bob consume a positive amount of good x in this solution?

- *Serving both markets.* If the monopolist serves both markets, by changing a unique price to Ann and Bob(not discriminating), it solves

$$\max_{p} \quad (a - \theta_B p)(p - c) + (a - \theta_A p)(p - c).$$

Taking first order condition with respect to p yields

$$a - 2\theta_B p + \theta_B c + a - 2\theta_A p + \theta_A c = 0,$$

rearranging

$$2a + c(\theta_A + \theta_B) = p \cdot 2(\theta_A + \theta_B)$$

which dividing over $2(\theta_A + \theta_B)$ yields an optimal price of

$$p^* = \frac{a}{\theta_B + \theta_A} + \frac{c}{2}.$$

- *Only serving Ann.* However, since Ann is willing to pay higher prices than Bob for the same quantities ($\theta_B > \theta_A$) then it may be better to charge a price above $\frac{a}{\theta_B}$. This cannot be captured in the above profit maximization problem since it induces Bob to demand a negative amount of the good. So we have to compare the above solution of p^* with the solution where the product is sold to Ann alone.

$$\max_p \quad (a - \theta_A p)(p - c).$$

Taking first order condition with respect to p yields

$$a - 2\theta_A \widehat{p} + \theta_A c = 0$$

rearranging, we obtain $a + \theta_A c = 2\theta_A \widehat{p}$. Dividing both sides by $2\theta_A$ yields a price of

$$\widehat{p} = \frac{a}{2\theta_A} + \frac{c}{2}.$$

- Figure 1 shows the two cases.

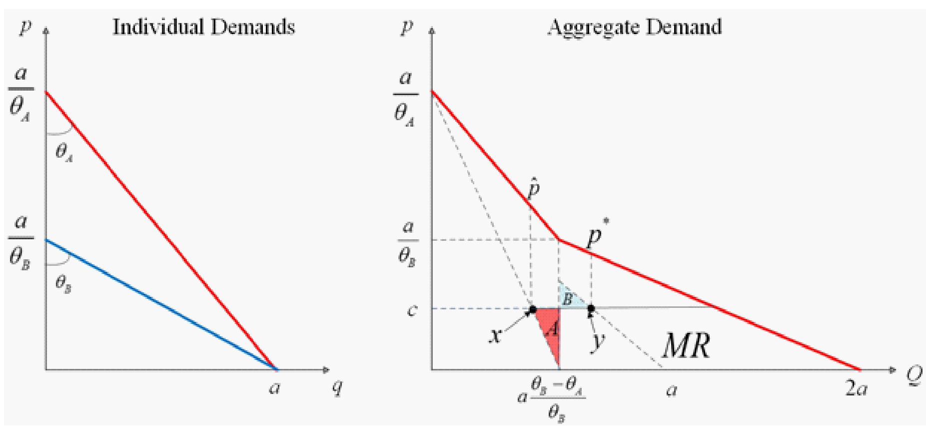

Figure 1. Individual and aggregate demand.

The aggregate demand curve must have a kink: for prices above the kink, only Ann buys positive units of the good, implying that aggregate demand coincides with Ann's demand. In contrast, for prices below the kink both Bob and Ann buy positive units of the good, implying that aggregate demand is equal to the sum of the demands of Bob and the demand of Ann.

- Due to the kink in the demand curve, the marginal revenue curve MR has a jump at that point (the dashed line). When equating the marginal cost c, to MR, we could have one or two solutions:

1. *When we have one solution*, it means that c cuts MR only to the left of

the discontinuity. If c cuts MR only to the left, then the optimal price is above $\frac{a}{\theta_B}$ and we only serve Ann.

2. *When we have two solutions*, then c cuts MR both to the left and to the right of the discontinuity, as shown in the figure by points x and y. When the monopolist moves from point x to y (i.e., from $\widehat{p}$ to p^*) it loses profits equal to triangle A, and gains profits equal to the triangle B.[1] The costs and benefits of such a move will determine if $\widehat{p}$ is optimal (only serving Ann) or if p^* is (both markets are served).

(c) If this monopolist has produced a total output level of X, what is the welfare-maximizing way to distribute it between Ann and Bob?

- Given quantity X, we maximize consumer surplus by determining the optimal distribution of output between Ann and Bob, q_A and q_B, that solves

$$\max_{q_A,q_B} \int_0^{q_B} p_B(x)\,dx + \int_0^{q_A} p_A(x)\,dx$$

$$\text{subject to} \quad q_B + q_A = X$$

and letting λ denote the Lagrange multiplier we get the first order conditions $p_B(q_B) = \lambda$ and $p_A(q_A) = \lambda$, which yield $p_B(q_B) = p_A(q_A)$, or

$$\frac{(a - q_B)}{\theta_B} = \frac{(a - q_A)}{\theta_A}.$$

Together with the constraint we solve for q_B and q_A, which yield output levels of

$$q_A = a + \frac{(x - 2a)\theta_A}{\theta_A + \theta_B} \quad \text{and} \quad q_B = \frac{a\theta_A + (x - a)\theta_B}{\theta_A + \theta_B}$$

(d) Suppose that the monopolist is allowed to discriminate. What prices does it charge?

- The discriminatory monopolist determines the price that it charges to Ann, p_A, and to Bob, p_B, that solves

$$\max_{p_B,p_A} \quad (a - \theta_B p_B)(p_B - c) + (a - \theta_A p_A)(p_A - c).$$

[1] Intuitively, at y the monopolist serves both customers (at a lower price) while at x it only serves the customer with the highest willingness to pay for the good (Ann in this case), selling fewer units but charging a higher price per unit.

Taking first order condition with respect to p_A yields

$$a - 2\theta_A p_A + \theta_A c = 0, \text{ or } p_A = \frac{a + \theta_A c}{2\theta_A},$$

and similarly, the first order condition with respect to p_B produces

$$a - 2\theta_B p_B + \theta_B c = 0, \text{ or } p_B = \frac{a + \theta_B c}{2\theta_B},$$

These prices yield quantities of

$$q_B = a - \theta_B \left(\frac{a + \theta_B c}{2\theta_B} \right) = \frac{a - c\theta_B}{2}, \text{ and}$$

$$q_A = a - \theta_A (\frac{a + \theta_A c}{2\theta_A}) = \frac{a - c\theta_A}{2}.$$

- Note that aggregate output under price discrimination is equal to the aggregate output *without* price discrimination if *both* markets were served

$$Q = a - \frac{(\theta_A + \theta_B) c}{2}$$

(e) In the case where the nondiscriminatory solution in (b) has positive consumption of good x by both Ann and Bob, does aggregate welfare rise or fall relative to the case in which discrimination is allowed? Relate your conclusion to your answer in (c).

- From part (c), we know that the welfare maximizing distribution for a given output is to set $p_B\left(q^B\right) = p_A\left(q^A\right)$, which is the case without price discrimination when both markets are served.
- Notice that under price discrimination $p_B\left(q^B\right) \neq p_A\left(q^A\right)$ if $\theta_B \neq \theta_A$, which implies that welfare is lower under price discrimination.

(f) What if the nondiscriminatory solution in (b) has only one type of consumer being served?

- If without price discrimination only Ann was served, then by allowing the monopolist to discriminate will open Bob's market (assuming that $c < \frac{a}{\theta_B}$) without changing the price to Ann. This means that there is added surplus from Bob's market, and that welfare under price discrimination is higher than when price discrimination is forbidden.