EconS 501 Midterm #2 - November 15th, 2020

Show all your work clearly and make sure you justify all your answers.
NAME_________________________

1. Consider a competitive market where the government is deciding to impose a per unit tax τ. Assume that the aggregate demand curve is $x(p)$ and aggregate supply curve $q(p)$. Assume that a partial equilibrium analysis is valid.

   (a) Show that an increase in τ decreases the price received by producers and increases the price paid by consumers less than proportionally.
   See slides 24-30 "Chapter 6"

   (b) Discuss when the tax is solely borne by consumers and when producers bear most of the tax burden.
   See slides 24-30 "Chapter 6"

   (c) Finally, explain when the consumption level maximizes the aggregate surplus $S(x)$. What is the effect of the per unit tax τ on $S(x)$?
   See slides 37-43 "Chapter 6"

2. Let us consider an individual that faces the following two simple lotteries over three outcomes, $L \equiv (b, a, c)$ and $L^{'} \equiv (z, z, a)$ for which she is indifferent, $L \sim L^{'}$, since

$$\max(b, a, c) = \max(z, z, a)$$

where probabilities satisfy: $a, b, c, z \in [0, 1]$, $a + b + c = 1$ and $a + z + a = 1$.

a. Please identify the individual's preference relation over these two lotteries and show that this preference relation does not satisfy the Independence Axiom. Do not use a numerical example.

This preference relation represents "*Extreme Preference for Certainty.*" Note that if

$$\max(b, a, c) = \max(z, z, a)$$

then it must be that $a > b, c, z$. Hence, we have that if $L \sim L^{'}$, by the IA the compound lotteries should also satisfy

$$\frac{1}{2}L + \frac{1}{2}L \sim \frac{1}{2}L + \frac{1}{2}L^{'}$$

or

$$L \sim \frac{1}{2}L + \frac{1}{2}L^{'}$$

but

$$\max(b, a, c) = \max(\frac{1}{2}b + \frac{1}{2}z, \frac{1}{2}a + \frac{1}{2}z, \frac{1}{2}c + \frac{1}{2}a)$$

In order to guarantee that this equality does not hold, and hence, IA is not satisfied we need to check three cases: (i) $a > \frac{1}{2}b + \frac{1}{2}z$, (ii) $a > \frac{1}{2}a + \frac{1}{2}z$ and $a > \frac{1}{2}c + \frac{1}{2}a$. We know that $a \geq b$, $a \geq c$ and $a \geq z$. Therefore, (i) is satisfied if

$$\begin{aligned} a &> \frac{1}{2}b + \frac{1}{2}z \\ 2a &> b + z \end{aligned}$$

case (ii) is satisfied if

$$\begin{aligned} a &> \frac{1}{2}a + \frac{1}{2}z \\ \frac{1}{2}a &> \frac{1}{2}z \\ a &> z \end{aligned}$$

which is true since $a > b, c, z$ Finally, case (iii)

$$\begin{aligned} a &> \frac{1}{2}c + \frac{1}{2}a \\ \frac{1}{2}a &> \frac{1}{2}c \\ a &> c \end{aligned}$$

which is also true since $a > b, c, z$
Therefore, the IA is not satisfied when $a > b, c, z$ and $2a > b + z$. In this case $\max(b, a, c) > \max(\frac{1}{2}b + \frac{1}{2}z, \frac{1}{2}a + \frac{1}{2}z, \frac{1}{2}c + \frac{1}{2}a)$, and hence, $L \succsim \frac{1}{2}L + \frac{1}{2}L'$.

b. Can the individual's preference relation over these two lotteries in part (a) be represented by an expected utility form linear in probabilities? Please justify your answer.

The preference relation needs to satisfy rationality, continuity and independence. But as proved in part (a) this preference relation does not satisfy the IA and hence it cannot be represented by an expected utility form.

c. Let us now consider a lottery with three possible outcomes: X, Y, and 0, where $X > Y > 0$ are monetary values. The individual faces two sets of choices: (i) in the first choice set there are two simple lotteries, $L_1 \equiv (a, b, a)$ and $L_1' \equiv (c, d, e)$ and (ii) in the second choice lotteries are $L_2 \equiv (a, b - d, d)$ and $L_2' \equiv (c, a, d + e)$; where probabilities satisfy: $a, b, c, d, e \in [0, 1]$, and the sum of probabilities for each lottery is equal to 1. Discuss the Allais' Paradox in this setting.

Consider lotteries $L_1 \equiv (a, b, a)$ and $L_1' \equiv (c, d, e)$ over three possible monetary outcomes:

$$\begin{array}{ccc} 1st & 2nd & 3rd \\ X & Y & 0 \end{array}$$

In addition, consider that the decision maker's preferences over lotteries have a EU form. Hence, $L_1 \succ L_1'$ implies

$$au_X + bu_Y + au_0 > cu_X + du_Y + eu_0$$

if we consider that $a = 0$, then we have

$$bu_Y > cu_X + du_Y + eu_0$$

By the IA, we can add $du_0 - du_Y$ on both sides

$$(du_0 - du_Y) + bu_Y > cu_X + du_Y + eu_0 + (du_0 - du_Y)$$

$$\underbrace{du_0 + u_Y(b-d)}_{EU \text{ of } L_2} > \underbrace{cu_X + u_0(d+e)}_{EU \text{ of } L_2'}$$

and hence

$$L_2 \succ L_2^{'}$$

However, empirical evidence has proved that the IA is violated since subjects usually prefer $L_1 \succ L_1^{'}$ and $L_2^{'} \succ L_2$ which violates the IA.

3. Consider an individual whose utility function is

$$u(x) = \exp(x)$$

where x stands for the wealth level of the individual, and $x > 0$.

(a) Is this individual a risk-averse or risk-lover?

- Differentiating the utility function with respect to x, we obtain

$$u'(x) \equiv \frac{du(x)}{dx} = \exp(x) > 0$$
$$u''(x) \equiv \frac{d^2u(x)}{dx^2} = \exp(x) > 0$$

Since the individual's utility is increasing at an increasing rate in x, his utility function is convex in wealth x, such that this individual is a risk-lover.

(b) Suppose the lottery gives this individual equal chances of obtaining wealth level x and $3x$. Find the certainty equivalent and probability premium of this lottery.

- The certainty equivalent is the amount that makes the individual indifferent to the expected utility of the lottery; that is, $u(CE) = E(u(x))$, which in our setting means

$$u(CE) = E(u(x)) = \frac{1}{2}\exp(x) + \frac{1}{2}\exp(3x)$$
$$\exp(CE) = \frac{1}{2}(\exp(x) + \exp(3x))$$

Solving for CE, yields

$$CE(x) = \ln(\exp(x) + \exp(3x)) - \ln 2.$$

- The probability premium is probability ε over fair odds that makes this individual indifferent to the utility of the expected value of the lottery; that is,

$$\left(\frac{1}{2} + \varepsilon\right)\exp(x) + \left(\frac{1}{2} - \varepsilon\right)\exp(3x) = \exp\left(\frac{x+3x}{2}\right)$$

that assigns more (less) probability to the worse (better) outcome because the individual loves the risk of losing over the expected value of the lottery.

Rearranging the above expression, we obtain

$$\varepsilon\left(\exp\left(x\right)-\exp\left(3x\right)\right)+\frac{\exp\left(x\right)+\exp\left(3x\right)}{2}=\exp\left(2x\right)$$

Simplifying, and solving for the probability premium $\varepsilon\left(x\right)$, yields

$$\varepsilon\left(x\right)=\frac{\exp\left(x\right)+\exp\left(3x\right)-2\exp\left(2x\right)}{2\left(\exp\left(3x\right)-\exp\left(x\right)\right)}$$

(c) Let $x=1$ in part (b). Find the certainty equivalent and probability premium.

- Inserting $x=1$ into the expression for the certainty equivalent, we obtain

$$CE\left(1\right)=\ln\left(\exp 1+\exp 3\right)-\ln 2=2.43$$

  implying that the individual needs to be paid \$2.43, which is above the expected value of the lottery at \$2, in order to make him indifferent to playing the lottery. Intuitively, we need to pay him more than the expected value of the lottery since he enjoys playing the lottery (recall that he is risk lover).
- Inserting $x=1$ into the expression for the probability premium, we find

$$\varepsilon\left(1\right)=\frac{\exp\left(1\right)+\exp\left(3\right)-2\exp\left(2\right)}{2\left(\exp\left(3\right)-\exp\left(1\right)\right)}=0.23$$

  implying that the individual needs to have probability $\frac{1}{2}+0.23=0.73$ of obtaining the worse outcome ($x=\$1$) and probability $\frac{1}{2}-0.23=0.27$ of obtaining the better outcome ($3x=\$3$) to make him indifferent to a sure amount of \$2. Intuitively, we need to make the worse (better) outcome more (less) likely, thus decreasing the expected utility from the lottery, so this risk-loving individual prefers the expected value of the lottery, \$2, than the less attractive lottery we just created. Recall that the opposite result applies when dealing with risk averse individuals, who need the worse (better) outcome to be less (more) likely.

4. Let us consider a market with two firms, Firm 1 and Firm 2, producing a homogeneous good. Both firms generate pollution during the production process, however, firm 1's pollution is αq_1 while firm 2's pollution is q_2, where $\alpha\in[0,1)$. Firm 1's marginal production costs is c_1 and it is strictly higher than that of firm 2, i.e., $c_1\geq c_2$. In addition, the social welfare is defined as follows

$$SW=CS+PS+T-Env$$

where CS is the consumer surplus, PS is the producer surplus, $T=t\left(\alpha q_1+q_2\right)$ is the tax revenue from emission fees, and $Env=\frac{d}{2}(\alpha q_1+q_2)^2$ is the environmental damage from the production of both firms, where $d>0$. Finally, the inverse demand function of firm $i=\{1,2\}$ is

$$p_i(q_i,q_j)=1-q_i-q_j \quad \text{where } j=\{1,2\} \text{ and } j\neq i.$$

where q_i denotes output.

(a) *No regulation.* Find equilibrium output levels when firms do not face emission fees. Identify the firm that produces more output, discuss your results.

- Firm 1 maximizes its profits as follows

$$\max_{q_i \geq 0} \; p_1(q_1, q_2)q_1 - c_1 q_1 = (1 - q_1 - q_2)\, q_1 - c_1 q_1$$

Differentiating with respect to q_1 yields

$$1 - 2q_1 - q_2 - c_1 = 0$$

Solving for q_1, we obtain firm 1's best response function

$$q_1(q_2) = \frac{1 - c_1}{2} - \frac{1}{2} q_2$$

which originates at $\frac{1-c_1}{2}$, and decreases in q_1 at a rate of $\frac{1}{2}$.

- Firm 2 maximizes its profits as follows

$$\max_{q_i \geq 0} \; p_2(q_1, q_2)q_2 - c_2 q_2 = (1 - q_1 - q_2)\, q_2 - c_2 q_2$$

Differentiating with respect to q_2 yields

$$1 - 2q_2 - q_1 - c_2 = 0$$

Solving for q_2, we obtain firm 2's best response function

$$q_2(q_1) = \frac{1 - c_2}{2} - \frac{1}{2} q_1$$

- We can then insert $q_2(q_1)$ into $q_1(q_2)$, as follows

$$q_1 = \frac{1 - c_1}{2} - \frac{1}{2}\left[\frac{1 - c_2}{2} - \frac{1}{2} q_1\right]$$

Solving for q_i, we obtain the equilibrium output under no regulation

$$q_1^* = \frac{1 - 2c_1 + c_2}{3}.$$

Since firms face different marginal production cost, $c_1 > c_2$, equilibrium output becomes $q_1^* = \frac{1-2c_1+c_2}{3}$ and $q_2^* = \frac{1-2c_2+c_1}{3}$. In this case firm2 produces more output than firm 1 by assumption.

1. b. *Regulation.* Find equilibrium output levels when firms face any emission fee t. When is $q_2^* > q_1^*$? Interpret your results with respect to α.

When firms are subject to an emission fee t per unit of output, the above maximization problem becomes

$$\max_{q_1 \geq 0} (1 - q_1 - q_2) q_1 - c_1 q_1 - t\alpha q_1$$

Solving for q_1, we obtain firm 1's best response function

$$q_1(q_2) = \frac{1 - c_B - \alpha t}{2} - \frac{1}{2} q_2.$$

- Firm 2 maximizes its profits as follows

$$\max_{q_i \geq 0} (1 - q_1 - q_2) q_2 - c_2 q_2 - t q_2$$

Differentiating with respect to q_2 yields

$$1 - 2q_2 - q_1 - c_2 - t = 0$$

Solving for q_2, we obtain firm 2's best response function

$$q_2(q_1) = \frac{1 - c_2 - t}{2} - \frac{1}{2} q_1$$

We can then insert $q_2(q_1)$ from part (b) into $q_1(q_2)$, as follows

$$q_1(q_2) = \frac{1 - c_1 - \alpha t}{2} - \frac{1}{2}[\frac{1 - c_2 - t}{2} - \frac{1}{2} q_1].$$

Since firms face different marginal cost and fee, $c_1 > c_2$ and t, equilibrium output simplifies to $q_1^* = \frac{1-2c_1+c_2-t(2\alpha-1)}{3}$ and $q_2^* = \frac{1-2c_2+c_1-t(2-\alpha)}{3}$. Note that $q_2^* > q_1^*$ if and only if $\alpha > 1 + \frac{c_2 - c_1}{t}$. Since $c_1 > c_2$ by assumption, then the second term is negative. Hence, the output level of firm 2 is greater than that of firm 1 if firm 1 polution is relatively high, that is, α is relatively high, and firm 1 is more negatively impacted by the burden imposed by the emission fee.

1. c. Identify the socially optimal output level for firm 1, q_1^{SO}, and for firm 2, q_2^{SO}.

The social planner chooses output levels q_1 and q_2 to maximize social welfare, as follows,

$$\begin{aligned}
&\max_{q_1, q_2 \geq 0} \; CS + \pi_1 + \pi_2 - Env + T \\
&\frac{1}{2}(q_1^2 + 2q_1 q_2 + q_2^2) + (1 - q_1 - q_2) q_1 - c_1 q_1 - t\alpha q_1 \\
&+ (1 - q_1 - q_2) q_2 - c_2 q_2 - t q_2 - \frac{d}{2}(\alpha q_1 + q_2)^2 + t(\alpha q_1 + q_2)
\end{aligned}$$

Differentiating with respect to q_1 we obtain

$$q_1 + q_2 + 1 - 2q_1 - q_2 - c_1 - q_2 - d\alpha(\alpha q_1 + q_2) = 0$$

Thus, solving for q_1, we find

$$q_1(q_2) = \frac{1 - c_1}{1 + d\alpha^2} - \frac{1 + d\alpha}{1 + d\alpha^2} q_2$$

Differentiating with respect to q_2 in the above maximization problem, we obtain

$$q_2 + q_1 - q_1 + 1 - 2q_1 - q_1 - c_2 - d(\alpha q_1 + q_2) = 0$$

Thus, solving for q_G, we find

$$q_2(q_1) = \frac{1-c_2}{1+d} - \frac{1+d\alpha}{1+d} q_1.$$

Inserting $q_2(q_1)$ into $q_1(q_2)$, and solving for q_1, yields the socially optimal output for the green firm

$$q_1^{SO} = \frac{c_2 - c_1 + d((1-c_1) - \alpha(1-c_2))}{d(\alpha-1)^2}$$

and the socially optimal output for the firm 2 is

$$q_2^{SO} = \frac{c_1 - c_2 + d(\alpha c_1 - \alpha(1-\alpha) - \alpha^2 c_2)}{d(\alpha-1)^2}$$

1. d. Find the socially optimal fees (t) that induce both firms to produce the socially optimal output levels found in part (c). Assume that $c_1 = c_2 = c$. When is the fee positive?

- The social optimal tax induces each type of firm to produce the socially optimal output q_i^{SO}. That is to say, we need to set $q_1(t) + q_2(t) = q_1^{SO} + q_2^{SO}$.Solving for t we obtain

$$\frac{1-c-t(2\alpha-1)}{3} + \frac{1-c-t(2-\alpha)}{3} = 1-c$$

$$t = \frac{c-1}{1+\alpha}$$

which is not positive since $c < 1$.

GOOD LUCK!