Homework # 8 - [Due on December 1st, 2021]

1. Consider a setting with N individuals, each of them simultaneously and independently deciding how many dollars to contribute to a public good. Assume that each individual has a Cobb-Douglas utility function $u(x_i, G) = x_i^{1-\alpha} G^{\alpha}$ where $G = \sum_{j=1}^{n} g_j$ denotes aggregate contributions and $\alpha \in (0,1)$ for all $i = 1, ..., N$. For simplicity, normalize the price of the public good.

   (a) Set up the utility maximization problem of agent i. Find the demand functions denoted $(x_i(\cdot), G(\cdot))$, for the private and public good.

   (b) Suppose that individuals are ranked according to wealth, whereby $\omega_1 \geq \omega_2 \geq ... \geq \omega_n$. Find conditions on ω_i and α for an equilibrium in which $g_2^* = ... = g_n^* = 0$ and agent 1 is the only contributor (only the richest individual contributes).

   (c) Let G_k denote aggregate donations in equilibrium when the total wealth W is divided equally among k individuals.

      i. Suppose first that we divide the wealth W among 2 individuals. Find aggregate donations in this case, G_2, and show that they are lower than aggregate donations when a single individual holds all the wealth, whereby $G_2 < G_1$.

      ii. More generally, suppose that the wealth is divided into k equal shares $\frac{W}{k}$ among k consumers. Compute the equilibrium value of G_k and show that $G_k \to 0$ when $k \to +\infty$. (The smallest amount of public production is supplied when everyone is a contributor).

2. Consider a monopolist facing inverse demand function $p(q) = 1 - q$; a supply function of $q = ax$, where x denotes the number of input that the monopolist hires (e.g., labor) and $a > 0$; and cost function $C(x) = bx + dx^2$, where $b, d > 0$, thus being increasing and convex in input units x.

   (a) Write down the monopolist's profit-maximization problem. Find the equilibrium values of the monopolist's input decision, and its output level.

   (b) Assume now that the firm operates in a perfectly competitive industry, where price equals marginal cost. Find in this context the equilibrium values of the monopolist's input decision, and its output level.

3. Consider a monopolist facing a linear inverse demand function $p(q) = a - q$ and a cost function $C(q) = (1-\alpha) q + \alpha q^2$, where $a > 2$ is the market size and $\alpha \in [0,1]$ denotes the firm's efficiency. In particular, when $\alpha = 0$, the monopolist's costs are linear, $C(q) = q$, while when $\alpha = 1$, its costs become convex, $C(q) = q^2$.

(a) Find the monopolist's profit-maximizing output q^m and equilibrium profits π^m.

(b) How does equilibrium output change with a and α? Explain.

(c) How do equilibrium profits change with a and α? Explain.

(d) Compare the equilibrium output found in part (a) to the perfectly competitive level q^*.

(e) *Numerical example.* Evaluate equilibrium output and profits in part (a) when $a = 3$ under three values of α: (i) $\alpha = 0$, (ii) $\alpha = 1/2$, and (iii) $\alpha = 1$. Compare and interpret your results.

4. Ann's total demand for good x is given by $x_A(p) = a - \theta_A p$, and Bob's total demand is $x_B(p) = a - \theta_B p$, where $\theta_A < \theta_B$. Intuitively, Bob's demand is more sensitive to a given increase in prices than Ann's. Alternatively, inverting these demand functions we obtain $p(q^A) = \frac{a}{\theta_A} - \frac{1}{\theta_A}x_A$ and $p(q^B) = \frac{a}{\theta_B} - \frac{1}{\theta_B}x_B$ for Ann and Bob, respectively. Hence, if $\theta_A < \theta_B$, then $\frac{a}{\theta_A} > \frac{a}{\theta_B}$, which ultimately implies that Ann's willingness to pay for the good is higher than Bob's. Finally, the (constant) marginal cost of production is $c > 0$.

(a) Suppose that the market for good x is competitive. Find the equilibrium quantity and price.

(b) Suppose, instead, that the firm is a monopolist. If this firm is prohibited from discriminating, what is its profit maximizing price? Under which conditions do Ann and Bob consume a positive amount of good x in this solution?

(c) If this monopolist has produced a total output level of X, what is the welfare-maximizing way to distribute it between Ann and Bob?

(d) Suppose that the monopolist is allowed to discriminate. What prices does it charge?

(e) In the case where the nondiscriminatory solution in (b) has positive consumption of good x by both Ann and Bob, does aggregate welfare rise or fall relative to the case in which discrimination is allowed? Relate your conclusion to your answer in (c).

(f) What if the nondiscriminatory solution in (b) has only one type of consumer being served?