Homework # 7 - [Due on November 10th, 2021]

1. Let us consider an individual with preferences $u(x,y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$, where x and y denote the amounts consumed of essential goods and non-essential goods, respectively. The prices of these goods are $p_x > 0$ and $p_y > 0$, respectively, and this individual's wealth is $w > 0$. The government needs to collect a large amount of money to finance the development of a vaccine, and considers two options:

   (a) introduce an income tax equivalent to 25% of individuals'wealth; or

   (b) charge a sales tax over the price of the non-essential good which would imply an increase in the price from p_y to $p_y(1+t)$, collecting the same dollar amount as with the income tax.

   Using the indirect utility function of this individual under option 1 (income tax) and option 2 (sales tax), explain which tax produces a smaller utility reduction.

- We need to calculate the effect of each type of tax on this individual's utility level, for a given total tax revenue. In order to find the individual's utility level in equilibrium (after he decides how to distribute his wealth among different goods) we must use this individual's indirect utility function. First, recall that the Walrasian demands in the context of a Cobb-Douglas utility function such as that given in this exercise are $x(p,w) = \frac{w}{2p_x}$ and $y(p,w) = \frac{w}{2p_y}$, respectively. Inserting these Walrasian demands into the individual's utility function we obtain the indirect utility function

$$v(p,w) = \frac{w}{2\sqrt{p_y p_z}}$$

Taking into account that the tax revenue must be 25% of the individual's wealth, i.e., $T = \frac{1}{4}w$, we need to find the increase in $p_y$that generates this tax revenue. If the price of the non-essential good increases as a result of a sales tax, i.e., $(1+t)p_y$, then Walrasian demand becomes $y(p,w) = \frac{w}{2(1+t)p_y}$, implying that tax revenue is

$$T = tp_y y(p,w) = tp_y \frac{w}{2(1+t)p_y} = \frac{t}{(1+t)}\frac{w}{2}$$

The condition that tax authorities generate the same revenue charging 25% of the individual's wealth, $\frac{1}{4}w$, and using a sales tax as described above, implies that

$$\frac{1}{4}w = \frac{t}{(1+t)}\frac{w}{2}$$

which, solving for t, yields a sales tax of $t = 1$.

- *Income tax.* If we use an income tax, then the individual's wealth after taxes becomes $w' = w - \frac{1}{4}w = \frac{3}{4}w$, inducing an after tax indirect utility function

$$v^{IncomeTax}(p, w) = \frac{w'}{2\sqrt{p_y p_z}} = \frac{3w}{8\sqrt{p_y p_z}}$$

- *Sales tax.* If, in contrast, we use a sales tax on the brown good, for an amount of $tp_y = p_y$, i.e., $p_y' = (1+t)p_y = 2p_y$, the new Walrasian demand for the brown good is $y' = \frac{w}{2 \times p_y}$, implying that the individual's indirect utility function in the case of the sales tax becomes

$$v^{SalesTax}(p, w) = \frac{w}{2\sqrt{p'_y p_z}} = \frac{w}{2\sqrt{2p_y p_z}}$$

Tax comparison. Comparing the indirect utility function of this individual under the income tax and the sales tax, we obtain that

$$v^{IncomeTax}(p, w) > v^{SalesTax}(p, w)$$

for all $y, z, w > 0$. That is, the individual prefers to bear a (revenue equivalent) income tax of 25% rather than a sale tax that increases the price of the brown good by $1.

2. Consider two consumers with utility functions over two goods, x_1 and x_2, given by

$$\begin{aligned} u_A &= \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) \quad \text{for consumer } A\text{, and} \\ u_B &= \log(x_1^B) + x_2^B - \frac{1}{2}\log(x_1^A) \quad \text{for consumer } B. \end{aligned}$$

where the consumption of good 1 by individual $i = \{A, B\}$ creates a negative externality on individual $j \neq i$ (see the third term, which enters negatively on each individual's utility function). For simplicity, consider that both individuals have the same wealth, m, and that the price for both goods is 1.

(a) *Unregulated equilibrium.* Set up consumer A's utility maximization problem, and determine his demand for goods 1 and 2, as x_1^A and x_2^A. Then operate similarly to find consumer B's demand for good 1 and 2, as x_1^B and x_2^B.

- Consumer A chooses x_1^A and x_2^A to solve

$$\max_{(x_1^A, x_2^A)} \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to } x_1^A + x_2^A = M$$

The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M - x_1^A - x_2^A),$$

which yields first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - x_1^A - x_2^A = 0$$

Solving for x_1^A, we obtain $\frac{1}{x_1^A} = 1$, i.e., $x_1^A = 1$, which implies $M - 1 - x_2^A = 0$, or $x_2^A = M - 1$. Hence, consumer A's optimal consumption is

$$x_1^A = 1 \text{ and } x_2^A = M - 1$$

A similar argument applies to consumer B,

$$x_1^B = 1 \text{ and } x_2^B = M - 1$$

(b) *Social optimum.* Calculate the socially optimal amounts of x_1^A, x_2^A, x_1^B and x_2^B, considering that the social planner maximizes a utilitarian social welfare function, namely, $W = U_A + U_B$.

- The socially optimal consumption in this case solves

$$\max_{(x_1^A, x_2^A)} U^A + U^B \text{ subject to } x_1^A + x_2^A = M \text{ and } x_1^B + x_2^B = M$$

The Lagrangian for this social planner's problem is

$$\mathcal{L} = \frac{1}{2}\log(x_1^A) + \frac{1}{2}\log(x_1^B) + x_2^A + x_2^B + \lambda^A(M - x_1^A - x_2^A) + \lambda^B(M - x_1^B - x_2^B)$$

Taking first-order conditions, we find the socially optimal consumption profile:

$$x_1^A = \frac{1}{2} \text{ and } x_2^A = M - \frac{1}{2}$$

$$x_1^B = \frac{1}{2} \text{ and } x_2^B = M - \frac{1}{2}$$

Intuitively, the social planner recommends a lower consumption of good 1 (the good that generates the negative externality), and an increase in the consumption of good 2, for both individuals.

(c) *Restoring efficiency.* Show that the social optimum you found in part (b) can be induced by a tax on good 1 (so the after-tax price becomes $1+t$) with the revenue returned equally to both consumers in a lump-sum transfer.

- With tax t^A placed on good 1 and with lump-sum transfer T^A, consumer A solves

$$\max_{(x_1^A, x_2^A)} \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$
$$\text{subject to } (1+t^A)x_1^A + x_2^A = M + T^A$$

where note that the price of good 1 increased from 1 to $(1+t^A)$, but this consumer also sees his wealth increase by the lump sum T^A. The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M + T^A - (1+t^A)x_1^A - x_2^A)$$

Taking first-order conditions, we obtain

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A(1+t^A) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M + T^A - (1+t^A)x_1^A - x_2^A = 0$$

Simultaneously solving for x_1^A and x_2^A, we find that consumer A's consumption bundles after introducing the tax become

$$x_1^A = \frac{1}{1+t^A} \quad \text{and} \quad x_2^A = M + T^A - 1$$

Similarly we find the optimal consumption of consumer B who pays tax t^Bon good 1 and receives T^Bas a lump-sum transfer:

$$x_1^B = \frac{1}{1+t^B} \quad \text{and} \quad x_2^B = M + T^B - 1$$

- *Comparison.* Comparing the optimal consumption levels found in part (b) with the equilibrium outcomes found in part (c), the tax imposed on any individual $i = A, B$ must hence satisfy

$$\frac{1}{2} = \frac{1}{1+t^i},$$

which would guarantee that equilibrium and socially optimal amounts coincide. Solving for the tax t^i yields $t^i = \$1$. Hence, by setting a tax of $t^i = \$1$ on the consumption of good 1, and returning the tax revenue to this individual in a lump-sum transfer, efficiency is restored, yielding a consumption

$$x_1^i = \frac{1}{1+1} = \frac{1}{2} \text{ of good 1,}$$

and

$$\begin{aligned} x_2^i &= M + T^i - 1 \\ &= M + \frac{1}{2} - 1 = M - \frac{1}{2} \text{ of good 2,} \end{aligned}$$

as described in the socially optimal amounts found in part (b).

3. Let us consider a market with two firms, Firm A and Firm B, producing a homogeneous good. However, Firm A generates more pollution than Firm B during the production process as explained below. Firm i's marginal production costs are given by c_i where $i = \{A, B\}$, where c_B is strictly higher than c_A. In addition, the social welfare function that the regulator uses to set emission fees on these firms is

$$SW = CS + PS + T - Env$$

where CS is the consumer surplus, PS is the producer surplus, $T = t\,(q_A + q_B)$ is the tax revenue from emission fees on both firms, and $Env = d_A(q_A)^2 + d_B(q_B)^2$ is the environmental damage from the production of both goods, where $d_A \geq d_B$. Finally, the inverse demand function of firm $i = \{A, B\}$ is

$$p_i(q_i, q_j) = 1 - q_i - q_j \text{ where } j = \{A, B\} \text{ and } j \neq i.$$

where q_i denotes output.

   (a) *No regulation.* Find equilibrium output levels when firms do not face emission fees. Interpret.

- Every firm i maximizes its profits as follows

$$\max_{q_i \geq 0} \; p_i(q_i, q_j)q_i - c_i q_i = (1 - q_i - q_j)\, q_i - c_i q_i$$

Differentiating with respect to q_i yields

$$1 - 2q_i - q_j - c_i = 0$$

Solving for q_i, we obtain firm i's best response function

$$q_i(q_j) = \frac{1 - c_i}{2} - \frac{1}{2} q_j$$

which originates at $\frac{1-c_i}{2}$, and decreases in q_j at a rate of $\frac{1}{2}$.

- Firm j's best response function is symmetric. We can then insert $q_j(q_i)$ into $q_i(q_j)$, as follows

$$q_i = \frac{1 - c_i}{2} - \frac{1}{2} \overbrace{\left(\frac{1 - c_j}{2} - \frac{1}{2} q_i \right)}^{q_j(q_i)}$$

Solving for q_i, we obtain the equilibrium output under no regulation

$$q_i^* = \frac{2(1 - c_i) - (1 - c_j)}{3}.$$

- Since firms face different marginal production cost, $c_B > c_A$, equilibrium output becomes $q_A^* = \frac{1 - 2c_A + c_B}{3}$ and $q_B^* = \frac{1 - 2c_B + c_A}{3}$. In this case firm A produces more output than firm B since $c_B > c_A$.

(b) *Regulation.* Find equilibrium output levels when firms face any emission fee t. Interpret.

- When the firm is subject to an emission fee t per unit of output, the above maximization problem becomes

$$\max_{q_i \geq 0} \; p_i\left(q_i, q_j\right) q_i - c_i q_i - t q_i = \left(1 - q_i - q_j\right) q_i - \left(c_i + t\right) q_i$$

where only the cost in the last term changed, from c_i to $(c_i + t)$. Following the same steps as in part (a), we find equilibrium output

$$q_i^* = \frac{2\left[1 - (c_i + t)\right] - \left[1 - (c_j + t)\right]}{3}.$$

which coincides with the expression found in part (b), except for the fact that each firm's marginal production cost is increased the emission fee.

- Since firms face different marginal cost and fee, $c_B > c_A$ and t, equilibrium output simplifies to $q_A^* = \frac{1-2c_A+c_B-t}{3}$ and $q_B^* = \frac{1-2c_B+c_A-t}{3}$. For each set of parameter values, every firm i sells fewer units when it is subject to the emission fee $t > 0$ than otherwise.

(c) Identify the socially optimal output level for firm A, q_A^{SO}, and for firm B, q_B^{SO}.

The social planner chooses output levels q_A and q_B to maximize social welfare, as follows,

$$\begin{aligned}&\max CS + PS + T - Env\\&\frac{1}{2}(q_A^2 + 2q_A q_B + q_B^2) + (1 - q_A - q_B)q_A - c_A q_A\\&+(1 - q_A - q_B)q_B - c_B q_B - d_A q_A^2 - d_B q_B^2\end{aligned}$$

Differentiating with respect to $q_A$we obtain

$$q_A + q_B + 1 - 2q_A - q_B - c_A - q_B - 2d_A q_A = 0$$

Further simplifying, yields

$$(1 + 2d_A)\, q_A = 1 - q_B - c_A$$

Thus, solving for q_A, we find

$$q_A(q_B) = \frac{1 - c_A}{1 + 2d_A} - \frac{1}{1 + 2d_A} q_B.$$

Differentiating with respect to q_B in the above maximization problem, we obtain a symmetric expression, $q_B(q_A)$. Inserting $q_A(q_B)$ into $q_B(q_A)$, and solving for q_A, yields the socially optimal output for the brown firm

$$q_A^{SO} = \frac{(1 + 2d_B)\,(1 - c_A) - (1 - c_B)}{(1 + 2d_A)\,(1 + 2d_B) - 1}$$

By symmetry, the socially optimal output for the green firm is $q_B^{SO} = \frac{(1+2d_A)(1-c_B)-(1-c_A)}{(1+2d_A)(1+2d_B)-1}$.

1. (d) Find the socially optimal fees t that induce firms to produce the socially optimal output levels found in part (c). Assume that $d_A = 2$ and $d_B = 0$,and $c_B = \frac{1}{4}$ and $c_A = 0$.

The social optimal tax induces each type of firm to produce the socially optimal output q_i^{SO}.

That is to say, we need to set $q_A(t) + q_B(t) = q_A^{SO} + q_B^{SO}$.Solving for t we obtain

$$\begin{aligned}\frac{1-2c_A+c_B-t}{3}+\frac{1-2c_B+c_A-t}{3} &= \frac{(1+2d_B)(1-c_A)-(1-c_B)}{(1+2d_A)(1+2d_B)-1} \\ &\quad +\frac{(1+2d_A)(1-c_B)-(1-c_A)}{(1+2d_A)(1+2d_B)-1} \\ t &= \frac{5}{16}\end{aligned}$$

which is always positive.