Homework # 6 - [Due on November 1st, 2021]

1. Consider a cumulative distribution function $F(x)$ which first-order stochastically dominates $G(x)$.

   (a) Show that the mean of x under $G(x)$, $\int x\, dG(x)$, cannot exceed that under $F(x)$, $\int x\, dF(x)$, i.e.,

   $$\int xdF(x) \geqslant \int xdG(x)$$

   - We know that distribution function $F(x)$ first-order stochastically dominates $G(x)$ if

     $$\int u(x)\, dF(x) \geqslant \int u(x)\, dG(x)$$

     Using the fact that the utility function is weakly increasing, we have

     $$\int xdF(x) \geqslant \int xdG(x)$$

   (b) Provide now an example where $\int xdF(x) \geqslant \int xdG(x)$ is satisfied, but $F(x)$ does not first order stochastically dominates $G(x)$.

   - Consider the two lotteries depicted in figure 1. The first one, $F(x)$, assigns 1/2 to monetary outcome \$2 and 1/2 to \$3. The second lottery, $G(x)$, evenly splits the probability weight that lottery $F(x)$ assigns to \$2 between \$1 and \$2 (each occurring with a probability of 1/4), and the probability weight of \$3 is also equally divided between \$3 and \$4.

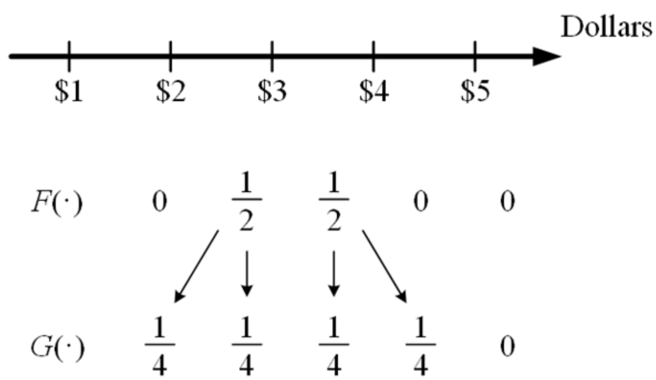

Figure 1. Lotteries $F(x)$ and.$G(x)$.

   - Thus, both lotteries $F(x)$ and $G(x)$ have the same expected value, $\frac{5}{2}$. However, neither $F(x)$ FOSD $G(x)$, nor $G(x)$ FOSD $F(x)$. In particular, depicting both probability distributions (see figure 2), one can easily observe that lottery $G(x)$ lies weakly above $F(x)$ for outcomes $x \leq \$3$, but lies below for monetary outcomes beyond that threshold.

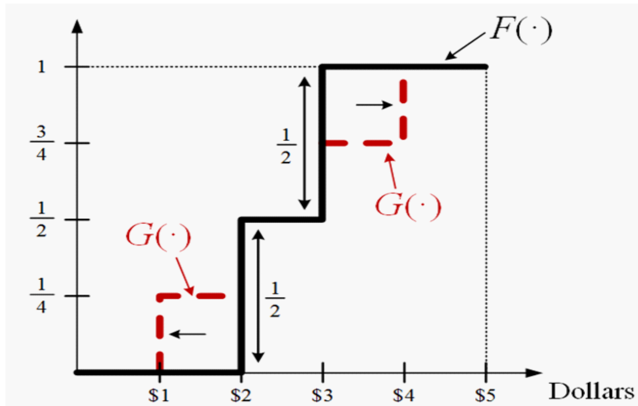

Figure 2. Lotteries $F(x)$ and.$G(x)$.

2. A tax is to be levied on a commodity bought and sold in a competitive market. Two possible forms of tax may be used: In one case, a *per unit* tax is levied, where an amount t is paid per unit bought or sold. In the other case, an *ad valorem* tax is levied, where the government collects a tax equal to τ times the amount the seller receives from the buyer. Assume that a partial equilibrium approach is valid.

   (a) Show that, with a per unit tax, the ultimate cost of the good to consumers and the amounts purchased are independent of whether the consumers or the producers pay the tax. As a guidance, let us use the following steps:

      1. *Consumers:* Let p^c be the competitive equilibrium price when the *consumer* pays the tax. Note that when the consumer pays the tax, he pays $p^c + t$ whereas the producer receives p^c. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the consumer pays the tax.

         - If the per unit tax t is levied on the consumer, then he pays $p+t$ for every unit of the good, and the demand at market price p becomes $x(p+t)$. The equilibrium market price p^c is determinded from equalizing demand and supply:

$$x(p^c + t) = q(p^c).$$

      2. *Producers:* Let p^p be the competitive equilibrium price when the *producer* pays the tax. Note that when the producer pays the tax, he receives $p^p - t$ whereas the consumer pays p^p. State the equality of the (generic) demand

and supply functions in the equilibrium of this competitive market when the producer pays the tax.

- On the other hand, if the per unit tax t is levied on the producer, then he collects $p - t$ from every unit of the good sold, and the supply at market price p becomes $q(p-t)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x(p^p) = q(p^p - t).$$

(b) Show that if an equilibrium price p solves your equality in part (a), then $p + t$ solves the equality in (b). Show that, as a consequence, equilibrium amounts are independent of whether consumers or producers pay the tax.

- It is easy to see that p solves the first equation if and only if $p + t$ solves the second one. Therefore, $p^p = p^c + t$, which is the ultimate cost of the good to consumers in both cases. The amount purchased in both cases is

$$x(p^p) = x(p^c + t).$$

(c) Show that the result in part (b) is not generally true with an ad valorem tax. In this case, which collection method leads to a higher cost to consumers? [*Hint:* Use the same steps as above, first for the consumer and then for the producer, but taking into account that now the tax increases the price to $(1+\tau)p$. Then, construct the excess demand function for the case of the consumer and the producer.]

- If the ad valorem tax τ is levied on the consumer, then he pays $(1+\tau)p$ for every unit of the good, and the demand at market price p becomes $x((1+\tau)p)$. The equilibrium market price p^c is determined from equalizing demand and supply:

$$x((1+\tau)p^c) = q(p^c).$$

On the other hand, if the ad valorem tax τ is levied on the producer, he receives $(1+\tau)p$ for every unit of the good sold, and the supply at market price p becomes $q((1-\tau)p)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x(p^p) = q((1-\tau)p^p).$$

Consider the excess demand function for this case:

$$z(p) = x(p) - q((1-\tau)p) \tag{1}$$

Since the demand curve $x(\cdot)$ is non-increasing and the supply curve $q(\cdot)$ is non-decreasing, $z(p)$ must be non-increasing. From (1) we have

$$\begin{aligned} z((1+\tau)p^c) &= x((1+\tau)p^c) - q((1-\tau)[(1+\tau)p^c]) = \\ &= x((1+\tau)p^c) - q((1-\tau^2)p^c) \geq \\ &\geq x((1+\tau)p^c) - q(p^c) = 0, \end{aligned}$$

where the inequality takes into account that $q(\cdot)$ is non-decreasing.

- Therefore, $z((1+\tau)p^c) \geq 0$ and $z(p^p) = 0$. Since $z(\cdot)$ is non-increasing, this implies that $(1+\tau)p^c \leq p^p$. In words, levying the ad valorem tax on consumers leads to a lower cost on consumers than levying the same tax on producers. (In the same way, it can be shown that levying the ad valorem tax on consumers leads to a higher price for producers than levying the same tax on producers).

(d) Are there any special cases in which the collection method is irrelevant with an ad valorem tax? [*Hint:* Think about cases in which the tax introduces the same wedge on consumers and producers (inelasticity). Then prove your statement by using the above argument on excess demand functions.]

- If the supply function $q(\cdot)$ is strictly increasing, the argument can be strengthened to obtain the strict inequality: $(1+\tau)p^c < p^p$. On the other hand, when the supply is perfectly inelastic, i.e., $q(p) = \overline{q}$ =constant, then yield

$$x((1+\tau)p^c) = \overline{q} = x(p^p),$$

and therefore $p^p = (1+\tau)p^c$. Here both taxes result in the same cost to consumers. However, producers still bear a higher burden when the tax is levied directly on them:

$$(1-\tau)p^p = (1-\tau)(1+\tau)p^c < p^c.$$

these prices are depicted in the next figure, where $x(p)$ reflects the demand function with no taxes and $x((1-\tau)p)$ represents the demand function with the ad valorem tax. While the inelastic supply curve guarantees that sales are unaffected by the tax (remaining at $\overline{q}$ units), the price that the producer receives drops from p^p to $(1-\tau)p^p$. Therefore, the two taxes are still not fully equivalent.

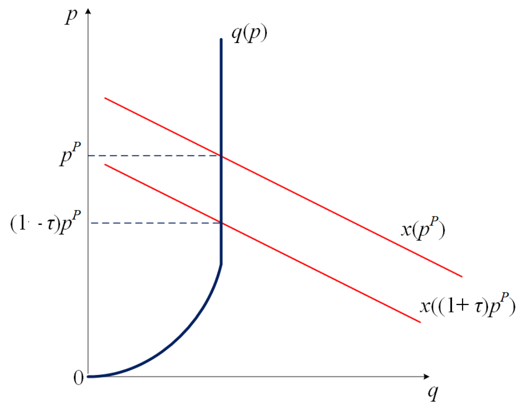

Figure 1. Introducing a tax.

- The intuition behind these results is simple: with a tax, there is always a wedge between the "consumer price"and the "producer price." Levying an ad valorem tax on the producer price, therefore, results in a higher tax burden (and a higher tax revenue) than levying the same percentage tax on consumers.

3. In our discussion of perfectly competitive markets, we considered that all firms produced a homogeneous good. However, our analysis can be easily extended to settings in which goods are heterogeneous. In particular, consider that every firm $i \in N$ faces a inverse demand function

$$p_i(q_i, q_{-i}) = \frac{\theta q_i^{\beta-1}}{\sum_{j=1}^{N} q_j^{\beta}}$$

where q_i denotes firm i's output, q_{-i} the output decisions of all other firms, i.e., $q_{-i} = (q_1, ..., q_{i-1}, q_{i+1}, ..., q_N)$, θ is a positive constant, and parameter $\beta \in (0, 1]$ captures the degree of substitutability. In addition, assume that every firm faces the same cost function $c(q_i) = F + cq_i$, where $F > 0$ denotes fixed costs and $c > 0$ represents marginal costs. Find the individual production level of every firm i, q_i^*, as a function of β. Interpret.

   - Every firm i's solves the following PMP

$$\max_{q_i} \frac{\theta q_i^{\beta-1}}{\sum_{j=1}^{N} q_j^{\beta}} q_i - (F + cq_i)$$

Taking first-order conditions with respect to q_i yields

$$\frac{\theta\left[\beta q_i^{\beta-1}\left(\sum_{j=1}^N q_j^\beta\right)-q_i^\beta\left(\beta q_i^{\beta-1}\right)\right]}{\left(\sum_{j=1}^N q_j^\beta\right)^2}-c=0$$

In a symmetric equilibrium, output levels satisfy $q_i^*=q^*$ for every firm $i\in N$, thus simplifying the above expression to

$$\frac{\theta\beta q^{2\beta-1}(N-1)}{N^2q^{2\beta}}-c=0$$

Solving for q^* yields the individual equilibrium output

$$q^*=\frac{\theta\beta(N-1)}{N^2c}$$

- *Comparative statics.* Differentiating q^* with respect to the substitutability parameter β we obtain

$$\frac{\partial q^*}{\partial\beta}=\frac{\theta(N-1)}{N^2c}>0$$

Hence, as goods become more differentiated (higher β), the equilibrium output level q^* rises. However, as more firms operate in this market, the increase in q^* becomes smaller since the derivative $\frac{\partial q^*}{\partial\beta}$ decreases in N, i.e.,

$$\frac{\partial\left(\frac{\partial q^*}{\partial\beta}\right)}{\partial N}=\frac{\theta N^2c-\theta(N-1)2Nc}{(N^2c)^2}<0.$$