Recitation #8 (10/22/2021)

1. Let G be the set of compound gambles over a finite set of deterministic payoffs $\{a_1, a_2, ...a_n\} \subset \mathbb{R}_+$. A decision maker's preference relation $\succsim$ over compound gambles can be represented by utility function $v : G \to \mathbb{R}$. Let $g \in G$, and let probability p_i be associated to the corresponding payoff a_i. Finally, consider that the decision maker's utility function $v(\cdot)$ is given by

$$v(g) = (1 + a_1)^{p_1} (1 + a_1)^{p_2} ...(1 + a_n)^{p_n} = \prod_{i=1}^{n} (1 + a_i)^{p_i}$$

   (a) Show that this is *not* a von Neumann-Morgenstern (vNM) utility function.

   - Since $v(g)$ is *not* linear in the probabilities, then $v(g)$ cannot be a vNM expected utility function, with general form

$$v(g) = \sum_{i=1}^{N} p_i u(a_i)$$

   (b) Show that the decision maker has the same preference relation as an expected utility maximizer with von-Neumann Morgenstern utility function

$$u(g) = \sum_{i=1}^{n} p_i \ln(1 + a_i).$$

   - Since $\ln(\cdot)$ is a monotonic transformation of $v(\cdot)$, both functions represent the same preference relation. Applying the monotonic transformation $u(g) = \ln[v(g)]$ to the original function $v(g)$, we obtain

$$\ln\left(\prod_{i=1}^{n} (1 + a_i)^{p_i}\right) = \sum_{i=1}^{n} p_i \ln(1 + a_i)$$

     which represents the initial preference relation over lotteries, and it is linear in the probabilities. Hence, it is a vNM utility function.

   (c) Assume now that the decision maker you considered in part (b) has utility function $u(w) = \ln(1 + w)$ over wealth $w \geqslant 0$. Evaluate his risk attitude (concavity in his utility function). Additionally, find the Arrow-Pratt coefficient of absolute risk aversion, $r_A(w, u)$. How does $r_A(w, u)$ change in wealth?

   - Given that $u(w) = \ln(1 + w)$, where $w \geqslant 0$, then the first and second order conditions with respect to w are

$$u'(w) = \frac{1}{1+w} > 0 \text{ and } u''(w) = -\frac{1}{(1+w)^2} < 0,$$

which implies that the utility function is concave, as depicted in figure 1, and that the decision maker is risk-averse.

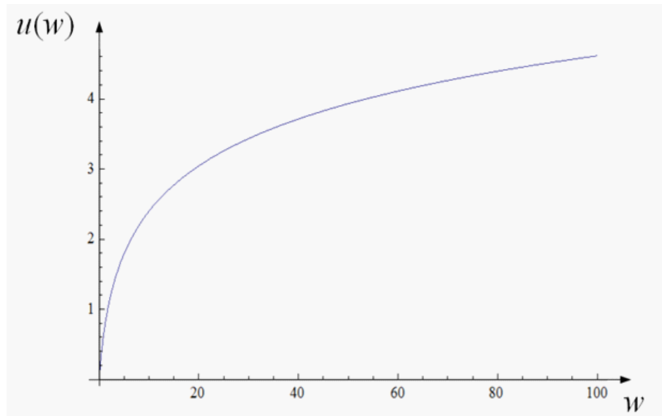

Figure 1. Utility function $u(w) = \ln(1+w)$

- Let us now obtain the Arrow-Pratt coefficient of absolute risk-aversion, $r_A(w,u)$, as follows

$$r_A(w,u) = -\frac{u''(w)}{u'(w)} = -\frac{-\frac{1}{(1+w)^2}}{\frac{1}{1+w}} = \frac{1}{1+w}$$

- Finally, we want to know how this coefficient of absolute risk aversion varies with wealth,

$$\frac{\partial r_A(w,u)}{\partial w} = -\frac{1}{(1+w)^2}$$

which is negative for all wealth levels $w \geqslant 0$. Hence, the agent becomes less risk-averse as he becomes more wealthy. Figure 2 illustrates this coefficient, $r_A(w,u)$, evaluated at different wealth levels.

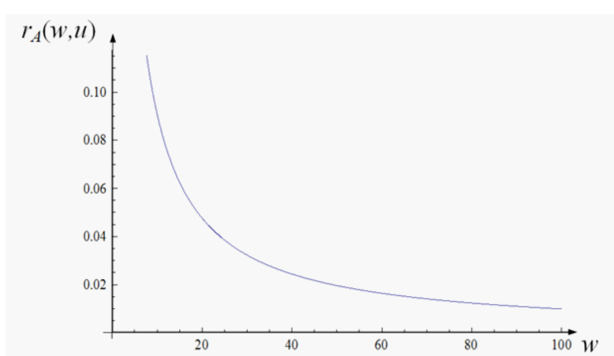

Figure 2. Coefficient of absolute risk aversion.

2. Consider the set of deterministic payoffs $\{a_1, a_2, ...a_n\} \subset \mathbb{R}_+$. Studies in regret-based decision making often consider the following utility function: first, define the highest deterministic payoff that could be reached in gamble g by using function

$$h(g) = \max\{a_k : k \in \{1, 2, ..., n\} \text{ and } p_k > 0\}.$$

Subtracting $h(g)$ from all deterministic outcomes and computing its expected value yields the utility level

$$v(g) = \sum_{i=1}^{n} p_i (a_i - h(g)) = \sum_{i=1}^{n} p_i a_i - h(g)$$

Intuitively, after event i realizes (which provides a payoff a_i to this individual), the "regretful" decision maker compares such monetary payoff with respect to the highest possible payoff he could have obtained from playing this lottery, $h(g)$. Utility functions of this type hence reflect "regret," as individuals experience a disutility from not receiving the highest possible monetary payoff in the lottery.[1]

(a) Compute the expected value of the following two gambles:

$$g^1 = \left(0, 1, 2; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ and } g^2 = \left(1, 4, 5; \frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$$

- First, note that $h(g^1) = \max\{0, 1, 2\}$ since all these events can occur with strictly positive probability in lottery g^1. Then, $h(g^1) = 2$, and therefore the individual's expected utility from playing the first gamble, g^1, is

$$v(g^1) = \frac{1}{3}(0-2) + \frac{1}{3}(1-2) + \frac{1}{3}(2-2) = -1$$

Similarly, we can find the expected utility from playing the second gamble, g^2. In particular, in this case the highest payoff of the lottery is $h(g^2) = \max\{1, 4, 5\} = 5$, implying that the expected utility from this gamble is

$$v(g^2) = \frac{1}{2}(1-5) + \frac{1}{3}(4-5) + \frac{1}{6}(5-5) = -\frac{7}{3}$$

Note that the individual experiences a lower expected utility from playing

[1]Recall that $h(g)$ is unaffected by probability p_i, as it describes the highest payoff receiving a positive probability.

the second than the first lottery. Intuitively, this happens because: (1) the distribution of payoffs in the second lottery is more spread than in the first lottery, and this makes the lower payoffs on the second gamble to be compared to a higher possible payoff $h(g^2)$; and (2) because the lowest payoffs on the second gamble are more likely than in the first and, as a consequence, the individual assigns a higher weight in the expected utility calculation to those monetary payoffs in which he is experiencing the biggest regret.

(b) Show that all deterministic outcomes (outcomes with probability 100%) yield the same utility level. That is, $v(a_1) = v(a_2) = ... = v(a_n)$.

- Let us represent by $v(a_i)$ the individual's utility level from a certain deterministic outcome a_i, i.e., $p_{a_i} = 1$. But if outcome a_i occurs with certainty, there is no potential regret. In particular, function $h(g)$ can only find the maximum among all outcomes of the lottery whose probability is strictly greater than zero. Since $p_{a_i} = 1$, then all other outcomes of the lottery receive probability zero, and hence

$$\begin{aligned} h(g) &= \max\{a_k : k \in \{1, 2, ..., n\} \text{ and } p_k > 0\} \\ &= \max\{a_i\} = a_i \end{aligned}$$

Therefore, the individual's expected utility becomes

$$v(a_i) = \sum_{i=1}^{n} p_i (a_i - h(g)) = 1(a_i - a_i) = 0$$

Thus, $v(a_1) = v(a_2) = ... = v(a_n) = 0$, regardless of the monetary payoff associated to outcome a_i. If there is just one event to be regretful about, my expected utility is zero!

(c) Show that the preference relation does not satisfy monotonicity if outcomes are deterministic.

- From the definition of monotonicity, we have that

$$(a_1, a_n; \alpha, 1-\alpha) \succsim (a_1, a_n; \beta, 1-\beta)$$

for all $\alpha, \beta \in [0, 1]$ if and only if $\alpha > \beta$. So if we make $\alpha = 1$ and $\beta = 0$, then the above condition on monotonicity becomes

$$(a_1, a_n; 1, 0) \succsim (a_1, a_n; 0, 1)$$

Then, clearly $a_1 \succsim a_n$ and $a_1 \not\precsim a_n$, which implies that $a_1 \succ a_n$.

- However, in part (b) we have shown that the individual's utility is the same (and equal to zero) when outcomes are deterministic. In other words, he is indifferent between gambles whose outcomes are deterministic, i.e., $a_1 \sim a_2 \sim ... \sim a_n$. But this contradicts that $a_1 \succ a_n$. Therefore, this "regretful" preference relation cannot satisfy monotonicity.