Midterm #1 EconS 501

Monday, October 11th, 2021

NAME: ____________________

Instructions. *Show all your work clearly and make sure you justify all your answers.*

1. Consider an individual consuming 2 goods, x_1 and x_2, with the following Stone-Geary utility function,

$$u(x_1, x_2) = \alpha \log(x_1 - \beta_1) + (1-\alpha)\log(x_2 - \beta_2)$$

where $\alpha \in [0,1]$, and β_1 (β_2) represents the subsistence level of consumption, for example, a minimum amount of food and water is required to keep the individual alive. Denote p_1 (p_2) the price of good 1 (2), and w the income of the individual, where $p_1, p_2, w > 0$.

(a) *UMP*. State the individual's utility maximization problem and find his Walrasian demand.

- The individual chooses x_1 and x_2 to solve the utility maximization problem,

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) = \alpha \log(x_1 - \beta_1) + (1-\alpha)\log(x_2 - \beta_2)$$

$$\text{subject to} \quad p_1 x_1 + p_2 x_2 = w$$

The Lagrangian associated with this utility maximization problem is

$$\mathcal{L} = \alpha \log(x_1 - \beta_1) + (1-\alpha)\log(x_2 - \beta_2) + \lambda(w - p_1 x_1 - p_2 x_2)$$

Differentiating the Lagrangian with respect to x_1, yields

$$\frac{\alpha}{x_1 - \beta_1} - \lambda p_1 \leq 0$$

Since $\lambda \geq \frac{\alpha}{p_1(x_1 - \beta_1)} > 0$ for all $0 < \alpha \leq 1$, we have that

$$\lambda = \frac{\alpha}{p_1(x_1 - \beta_1)}$$

Differentiating now the Lagrangian with respect to x_2, we find

$$\frac{1-\alpha}{x_2 - \beta_2} - \lambda p_2 \leq 0$$

Since $\lambda \geq \frac{1-\alpha}{p_2(x_2 - \beta_2)} > 0$ for all $0 \leq \alpha < 1$, we obtain that

$$\lambda = \frac{1-\alpha}{p_2(x_2 - \beta_2)}$$

Differentiating the Lagrangian with respect to λ, yields

$$w - p_1 x_1 - p_2 x_2 \geq 0$$

Since $\lambda > 0$, the budget constraint binds, yielding

$$x_2 = \frac{w - p_1 x_1}{p_2}$$

Therefore, for any $0 < \alpha < 1$, we have an interior solution, where the individual consumes positive units of both goods. Equating the expressions for λ,

$$\frac{\alpha}{p_1 (x_1 - \beta_1)} = \frac{1-\alpha}{p_2 (x_2 - \beta_2)}$$

we obtain the price ratio equal to the marginal rate of substitution, as follows,

$$\frac{p_1}{p_2} = \frac{\alpha (x_2 - \beta_2)}{(1-\alpha)(x_1 - \beta_1)}$$

which is rearranged to yield

$$x_2 = \beta_2 + \frac{(1-\alpha) p_1}{\alpha p_2} (x_1 - \beta_1) .$$

Equating the expressions for x_2,

$$\frac{w - p_1 x_1}{p_2} = \beta_2 + \frac{(1-\alpha) p_1}{\alpha p_2} (x_1 - \beta_1)$$

which simplifies that

$$\alpha w - \alpha p_1 x_1 = \alpha p_2 \beta_2 + (1-\alpha)(p_1 x_1 - p_1 \beta_1) .$$

Rearranging, the Walrasian demand function for good x_1 becomes

$$x_1 (p_1, p_2, w) = \beta_1 + \frac{\alpha (w - p_1 \beta_1 - p_2 \beta_2)}{p_1}$$

Similarly, the Walrasian demand function for good x_2 becomes

$$x_2 (p_1, p_2, w) = \beta_2 + \frac{(1-\alpha)(w - p_1 \beta_1 - p_2 \beta_2)}{p_2}$$

(b) Find the value function, and show that it is homogeneous of degree zero. Interpret.

- Substituting the demand in part (a) into the individual's utility function, we find

$$\begin{aligned} v(p_1, p_2, w) &= \alpha \log \underbrace{\frac{\alpha (w - p_1 \beta_1 - p_2 \beta_2)}{p_1}}_{x_1 - \beta_1} + (1-\alpha) \log \underbrace{\frac{(1-\alpha)(w - p_1 \beta_1 - p_2 \beta_2)}{p_2}}_{x_2 - \beta_2} \\ &= \alpha \log \frac{\alpha}{p_1} + (1-\alpha) \log \frac{1-\alpha}{p_2} + \log (w - p_1 \beta_1 - p_2 \beta_2) \end{aligned}$$

- Consider a common increase in prices and income by a common factor μ, where

$$\begin{aligned} v(\mu p_1, \mu p_2, \mu w) &= \alpha \log \frac{\alpha}{\mu p_1} + (1-\alpha)\log \frac{1-\alpha}{\mu p_2} + \log(\mu w - \mu p_1 \beta_1 - \mu p_2 \beta_2) \\ &= \alpha \log \frac{\alpha}{p_1} + (1-\alpha)\log \frac{1-\alpha}{p_2} + \log(w - p_1 \beta_1 - p_2 \beta_2) \\ &= v(p_1, p_2, w). \end{aligned}$$

  Therefore, the value function is homogeneous of degree zero. When prices and income are increased by a common factor λ, the individual's real income is unaffected, thereby consuming the same bundle to attain the same utility level.

(c) How does the utility level you identified in part (b) change in β_1, p_1, and w? Explain.

- Differentiating the value function with respect to β_1, we obtain that

$$\frac{\partial v(p_1, p_2, w)}{\partial \beta_1} = -\frac{p_1}{w - p_1\beta_1 - p_2\beta_2} < 0$$

  As the subsistence level of good 1 increases (higher β_1), the individual has less disposable income (that is, $w - p_1\beta_1 - p_2\beta_2$ decreases), so his utility decreases.

- Differentiating the value function with respect to p_1, we have

$$\frac{\partial v(p_1, p_2, w)}{\partial p_1} = -\frac{\alpha}{p_1} - \frac{\beta_1}{w - p_1\beta_1 - p_2\beta_2} < 0$$

  Intuitively, as good 1 becomes more costly (higher p_1), the individual needs to spend more into buying the subsistence level of good 1, so his utility decreases.

- Finally, differentiating the value function with respect to w, yields

$$\frac{\partial v(p_1, p_2, w)}{\partial w} = \frac{1}{w - p_1\beta_1 - p_2\beta_2} > 0$$

  Therefore, as the individual has a higher income, he can consume more of both goods, ultimately increasing his utility.

(d) Show that the Roy's Identity holds.

- The Roy's Identity requires that

$$-\frac{\frac{\partial v(p_1,p_2,w)}{\partial p_1}}{\frac{\partial v(p_1,p_2,w)}{\partial w}} = x_1(p_1, p_2, w) \quad \text{and} \quad -\frac{\frac{\partial v(p_1,p_2,w)}{\partial p_2}}{\frac{\partial v(p_1,p_2,w)}{\partial w}} = x_2(p_1, p_2, w)$$

- *Good 1.* To check the first identity, note that from the results in part (c), we have that

$$\begin{aligned} -\frac{\frac{\partial v(p_1,p_2,w)}{\partial p_1}}{\frac{\partial v(p_1,p_2,w)}{\partial w}} &= -\frac{-\frac{\alpha}{p_1} - \frac{\beta_1}{w-p_1\beta_1-p_2\beta_2}}{\frac{1}{w-p_1\beta_1-p_2\beta_2}} \\ &= \beta_1 + \frac{\alpha(w - p_1\beta_1 - p_2\beta_2)}{p_1} \\ &= x_1(p_1, p_2, w) \end{aligned}$$

  so Roy's Identity holds for good 1.

- *Good 2.* Next, differentiating the value function with respect to p_2, we find that
$$\frac{\partial v(p_1,p_2,w)}{\partial p_2} = -\frac{1-\alpha}{p_2} - \frac{\beta_2}{w - p_1\beta_1 - p_2\beta_2}$$
This allows us to check the second identity, as follows
$$\begin{aligned} -\frac{\frac{\partial v(p_1,p_2,w)}{\partial p_2}}{\frac{\partial v(p_1,p_2,w)}{\partial w}} &= -\frac{-\frac{1-\alpha}{p_2} - \frac{\beta_2}{w-p_1\beta_1-p_2\beta_2}}{\frac{1}{w-p_1\beta_1-p_2\beta_2}} \\ &= \beta_2 + \frac{(1-\alpha)(w-p_1\beta_1-p_2\beta_2)}{p_2} \\ &= x_2(p_1,p_2,w) \end{aligned}$$
Therefore, we conclude that the Roy's Identity holds for good 2 as well.

2. Consider an individual with utility function
$$u(x_1,x_2) = \frac{1}{2}\log x_1 + \frac{1}{2}\log x_2$$
for good 1 and 2. Prices for each good i, p_i, is strictly positive.

   (a) If the individual seeks a minimal utility level $u > 0$, find his Hicksian demand for each good.

   - The individual's expenditure minimization problem is
$$\min_{x_1,x_2 \geq 0} \quad p_1x_1 + p_2x_2$$
$$\text{subject to } \frac{1}{2}\log x_1 + \frac{1}{2}\log x_2 \geq u$$
His Lagrangian function then becomes
$$\mathbb{L} = p_1x_1 + p_2x_2 + \lambda\left(u - \frac{1}{2}\log x_1 - \frac{1}{2}\log x_2\right)$$
Assuming interior solutions, where $x_1, x_2 > 0$, we can use the tangency condition $MRS_{12} = \frac{p_1}{p_2}$, which in this case is
$$\frac{x_2}{x_1} = \frac{p_1}{p_2}, \text{ or } x_2 = \frac{p_1}{p_2}x_1$$
Substituting $x_2 = \frac{p_1}{p_2}x_1$ into the utility constraint, we find
$$\frac{1}{2}\log(x_1) + \frac{1}{2}\log\overbrace{\left(\frac{p_1}{p_2}x_1\right)}^{x_2} = u$$
which is a function of x_1 alone. Rearranging, we obtain
$$\log(x_1) = u - \log\sqrt{\frac{p_1}{p_2}}$$

and solving for x_1, yields the Hicksian demand for good 1,

$$h_1(p,u) = \exp(u)\sqrt{\frac{p_2}{p_1}}.$$

Inserting this result into the tangency condition, $x_2 = \frac{p_1}{p_2}x_1$, we find the Hicksian demand for good 2, as follows,

$$\begin{aligned} h_2(p,u) &= \frac{p_1}{p_2}\overbrace{\left(\exp(u)\sqrt{\frac{p_2}{p_1}}\right)}^{h_1(p,u)} \\ &= \exp(u)\sqrt{\frac{p_1}{p_2}}. \end{aligned}$$

(b) Show that Hicksian demands are homogeneous of degree zero in p.

- To check for homogeneity in prices, we increase all prices by a common constant $\alpha > 0$, to show that Hicksian demands are unaffected by this common increase in all prices, that is,

$$\begin{aligned} h_1(\alpha p,u) &= \exp(u)\sqrt{\frac{\alpha p_2}{\alpha p_1}} = \exp(u)\sqrt{\frac{p_2}{p_1}} = h_1(p,u) \\ h_2(\alpha p,u) &= \exp(u)\sqrt{\frac{\alpha p_1}{\alpha p_2}} = \exp(u)\sqrt{\frac{p_1}{p_2}} = h_2(p,u) \end{aligned}$$

  Therefore, Hicksian demand is homogeneous of degree zero in prices.

(c) Why is the interior solution you found in part (a) unique? A verbal discussion suffices.

- In the case of corner solutions, either $x_1 = 0$ or $x_2 = 0$, implying that the resulting utility approaches $-\infty$, thus falling below the minimal utility level $u > 0$ that the individual seeks to reach. Therefore, the interior solution $(h_1(p,u), h_2(p,u))$ we found in part (a) is the only solution to the expenditure minimization problem.

(d) Find the expenditure function $e(p,u)$.

- The expenditure function is

$$\begin{aligned} e(p,u) &= p_1 h_1(p,u) + p_2 h_2(p,u) \\ &= p_1\left(\exp(u)\sqrt{\frac{p_2}{p_1}}\right) + p_2\left(\exp(u)\sqrt{\frac{p_1}{p_2}}\right) \\ &= 2\exp(u)\sqrt{p_1 p_2} \end{aligned}$$

(e) Show that if the compensated law of demand (CLD) holds, good 1 and good 2 must be *net* substitutes. [*Hint*: Apply the Euler's Theorem.]

- By Euler's Theorem, differentiate $h(\alpha p,u) = h(p,u)$ (from the Hicksian demand being homogeneous of degree zero in prices) with respect to the common constant α, to obtain

$$\frac{\partial h(\alpha p,u)}{\partial \alpha} = \frac{\partial h(p,u)}{\partial \alpha}$$

ellaborating this derivative, we find

$$\frac{\partial h(\alpha p,u)}{\partial(\alpha p_1)}\frac{\partial(\alpha p_1)}{\alpha}+\frac{\partial h(\alpha p,u)}{\partial(\alpha p_2)}\frac{\partial(\alpha p_2)}{\alpha}=0$$

which, in the case that $\alpha = 1$, simplifies to

$$\frac{\partial h(p,u)}{\partial p_1}p_1+\frac{\partial h(p,u)}{\partial p_2}p_2=0$$

or

$$\frac{\partial h(p,u)}{\partial p_1}p_1=-\frac{\partial h(p,u)}{\partial p_2}p_2$$

For good 1, CLD implies $\frac{\partial h_1(p,u)}{\partial p_1}<0$, yielding $\frac{\partial h_1(p,u)}{\partial p_2}>0$ as required by Euler's Theorem. A similar argument applies for the Hicksian demand of good 2. Thus, goods 1 and 2 are *net* substitutes.

(f) Show that the Shephard's Lemma holds for the expenditure function $e(p,u)$ that you found in part (d).

- Without loss of generality, we consider a small change in p_1 holding p_2 constant. Results are analogous for a small change in p_2 holding p_1 constant.
- Differentiating the expenditure function with respect to p_1, yields

$$\begin{aligned}\frac{\partial e(p,u)}{\partial p_1}&=\frac{\partial\left[p_1h_1(p,u)+p_2h_2(p,u)\right]}{\partial p_1}\\&=h_1(p,u)+p_1\underbrace{\frac{\partial h_1(p,u)}{\partial p_1}}_{=0\text{ for }\partial p_1\to 0}+p_2\underbrace{\frac{\partial h_2(p,u)}{\partial p_1}}_{=0\text{ for }\partial p_1\to 0}\\&=h_1(p,u)\end{aligned}$$

- Alternatively, we can take the derivative of expenditure function $e(p,u)$ with respect to p, obtaining

$$\begin{aligned}\frac{\partial e(p,u)}{\partial p_1}&=\frac{\partial\left[2\exp(u)\sqrt{p_1p_2}\right]}{\partial p_1}\\&=\exp(u)\sqrt{\frac{p_2}{p_1}}\\&=h_1(p,u)\end{aligned}$$

3. Consider a firm using $N\geq 2$ inputs to produce an output, with Cobb-Douglas production function

$$f(x)=\sum_{i=1}^{N}\alpha_i\log x_i$$

where $\alpha_i\geq 0$ and $\sum_{i=1}^{N}\alpha_i=1$. The firm is a price taker in both the input and output markets, every unit of input x_i can be purchased at a price w_i and every unit of output can be sold at a price of p.

(a) Set up the firm's profit maximization problem and find its equilibrium input and output. How are the equilibrium input and output affected by α_i, w_i, and p? Explain.

- The firm chooses the input vector $x = (x_1, x_2, ..., x_N)$ to solve the following profit maximization problem,

$$\max_{x \geq 0} \pi(p, w) = pf(x) - wx$$

Taking first order conditions with respect to every input x_i, and assuming interior solutions, yields

$$\frac{p\alpha_i}{x_i} = w_i \text{ for every } i.$$

After rearranging, we find the following factor demand function,

$$x_i(p, w) = \frac{p\alpha_i}{w_i}$$

which is increasing in output price p and factor share α_i, but decreasing in input price w_i. Intuitively, the firm uses more units of x_i when output can be sold at a higher price p or the production process uses this input more intensively (higher α_i), but fewer units when this input becomes more costly to purchase (higher w_i).

- Substituting equilibrium factor demands into the production function, we find

$$f(p, w) = \sum_{i=1}^{N} \alpha_i \log \underbrace{\left(\frac{p\alpha_i}{w_i}\right)}_{x_i}$$

which is, as expected, increasing in the output price p but decreasing in input price w_i. Finally, differentiating the output function with respect to α_i, we obtain that

$$\frac{\partial f(p, w)}{\partial \alpha_i} = \log \frac{p\alpha_i}{w_i} + 1$$

which is positive if and only if

$$\alpha_i > \frac{w_i}{p} \exp(-1).$$

Intuitively, when factor x_i is used relatively intensively in the production process, further increasing its share (higher α_i) will lead the firm to produce more units of output.

(b) Are the input and output functions homogeneous? If yes, of what degree? Interpret.

- When input and output prices are increased by a common factor λ, we find that

$$x_i(\lambda p, \lambda w) = \frac{\lambda p\alpha_i}{\lambda w_i} = \frac{p\alpha_i}{w_i} = x_i(p, w)$$

so the factor demand function is homogeneous of degree zero. Intuitively, when relative prices of input and output are unchanged, factor use is unaffected.

- Applying the same proportional increase in prices to the output function, yields

$$f(\lambda p, \lambda w) = \sum_{i=1}^{N} \alpha_i \log \frac{\lambda p \alpha_i}{\lambda w_i} = \sum_{i=1}^{N} \alpha_i \log \frac{p \alpha_i}{w_i} = f(p, w)$$

so the output function is also homogeneous of degree zero. From the previous result, the firm uses the same amount of inputs after the price change (as relative prices were not affected), which implies that the firm keeps producing the same units of output.

(c) Find the marginal rate of technical substitution.

- The marginal rate of technical substitution between any two inputs x_i and x_j is

$$MRTS_{ij} = \frac{\frac{\partial f(x)}{\partial x_j}}{\frac{\partial f(x)}{\partial x_i}} = \frac{\alpha_j x_i}{\alpha_i x_j}$$

As in a Cobb-Douglas production function with only two inputs, this MRTS is decreasing in input x_j indicating that, as the firm uses more units of x_j, it needs more units of x_j when decreasing x_i by one unit while keeping total output unaffected.

(d) Find the elasticity of substitution.

- Taking natural logs on both sides of the MRTS in part (c), we find that

$$\log MRTS_{ij} = \log \frac{\alpha_j}{\alpha_i} + \log \frac{x_i}{x_j}$$

and, rearranging, we obtain

$$\log MRTS_{ij} = -\log \frac{\alpha_i}{\alpha_j} + \log \frac{x_i}{x_j}$$

which can be expressed as

$$\log \frac{x_i}{x_j} = \log MRTS_{ij} + \log \frac{\alpha_i}{\alpha_j}$$

- Therefore, the elasticity of substitution becomes

$$\sigma = \frac{\partial \log \frac{x_i}{x_j}}{\partial \log MRTS_{ij}} = 1$$

which is a constant, implying that the firm's ability to substitute any two inputs x_i and x_j is independent of its output level. This is a common property in Cobb-Douglas production functions.

4. Discuss the relationship between homogeneity and homotheticity and show that homotheticity does not imply homogeneity.

See page 42-43 Munoz-Garcia's Textbook.