Homework #5 (Due on October 25th, 2021)

1. Consider an individual with the following utility function, where x denotes income.

$$u(x) = \begin{cases} 2x & \text{if} \quad x \leq \frac{5}{2} \\ \frac{5}{2} + x & \text{if} \quad x > \frac{5}{2} \end{cases}$$

(a) Depict the utility function with $u(x)$ on the vertical axis and income, x, on the horizontal axis. Show that this individual is (weakly) risk averse.

- This individual's utility can be expressed as the minimum of $2x$ and $\frac{5}{2} + x$. In particular $2x$ is the minimum of these linear functions for all x satisfying $2x \leqslant \frac{5}{2} + x$, or $x \leq \frac{5}{2}$; and $\frac{5}{2} + x$ is the minimum of both linear functions for $x > \frac{5}{2}$. Hence, the function can be represented $u(x) = \min\left\{2x, \frac{5}{2} + x\right\}$, as figure 1 depicts. Specially, $\min\left\{2x, \frac{5}{2} + x\right\}$ considers $2x$ for the interval $x \leqslant \frac{5}{2}$, and $\frac{5}{2} + x$ for value of x beyond that cutoff. As the figure illustrates, the function $u(x) = \min\left\{2x, \frac{5}{2} + x\right\}$, depicted the lower envelope of the lines $\frac{5}{2} + x$ and $\frac{5}{2}$, and it is a (weakly) concave function.

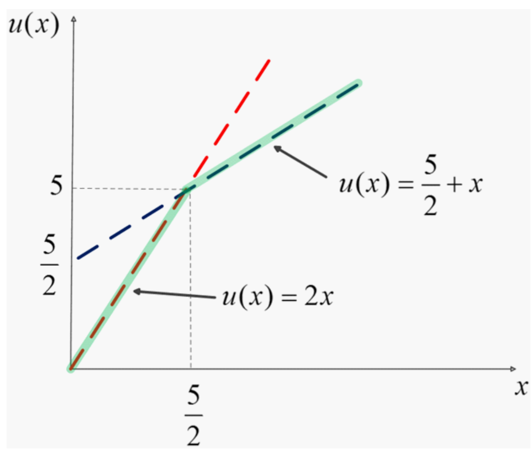

Figure 1. $u(x) = \min\left\{2x, \dfrac{5}{2} + x\right\}$

(b) Suppose that there are three states of the world, each equally likely. There are two assets, x and y. The asset x is the random variable with payoffs $(1, 5, 9)$ and the asset y is the random variable with payoffs $(2, 3, 10)$. (Note that assets specify a payoff triple, to indicate the payoff arising in each of the three equally likely states of the world.) Calculate the expected utility of asset x and of asset y. Which asset, hence, would be preferred by this individual, if both of them were offered at the same price?

- Let us first find the expected utility of asset x,

$$\begin{aligned} EU(x) &= \frac{1}{3}\min\left\{2\times 1, \frac{5}{2}+1\right\} + \frac{1}{3}\min\left\{2\times 5, \frac{5}{2}+5\right\} \\ &+\frac{1}{3}\min\left\{9\times 1, \frac{5}{2}+9\right\} \end{aligned}$$

where the first term represents that the first outcome occurs, yielding a payoff of \$9, and the second (third) term reflect the second (third) outcome with payoff \$5 (\$9, respectively). Simplifying this expression, we obtain

$$= \frac{1}{3}\times 2 + \frac{1}{3}\times\frac{15}{2} + \frac{1}{3}\times\frac{23}{2} = 7$$

And similarly for the expected utility of asset y,

$$\begin{aligned} EU(y) &= \frac{1}{3}\min\left\{2\times 2, \frac{5}{2}+2\right\} + \frac{1}{3}\min\left\{2\times 3, \frac{5}{2}+3\right\} \\ &+\frac{1}{3}\min\left\{9\times 10, \frac{5}{2}+10\right\} \\ &= \frac{1}{3}\times 4 + \frac{1}{3}\times\frac{11}{2} + \frac{1}{3}\times\frac{25}{2} = \frac{22}{3} \simeq 7.33 \end{aligned}$$

Therefore, $EU(x) < EU(y)$, making asset y to be preferred by the individual, if both assets were offered at the same price.

(c) Calculate the expected *value* of each asset (you previously found the expected *utility*). Calculate the variance of both assets. Which asset would be chosen by this individual if he were variance averse?

- The expected value of each asset is

$$\begin{aligned} E(x) &= \frac{1}{3}1 + \frac{1}{3}5 + \frac{1}{3}9 = 5 \\ E(y) &= \frac{1}{3}2 + \frac{1}{3}3 + \frac{1}{3}10 = 5 \end{aligned}$$

Hence, both assets have the same expected value. Let us now find their variance. For convenience, we use the formula $Var(x) = E(x^2) - E(x)^2$. We already know $E(x)$ and $E(y)$, let us then find $E(x^2)$ for asset x and $E(y^2)$ for asset y,

$$\begin{aligned} E(x^2) &= \frac{1}{3}(1)^2 + \frac{1}{3}(5)^2 + \frac{1}{3}(9)^2 = \frac{107}{3} \\ E(y^2) &= \frac{1}{3}(2)^2 + \frac{1}{3}(3)^2 + \frac{1}{3}(10)^2 = \frac{113}{3} \end{aligned}$$

Therefore, we can use this information, together with the expected values of the assets, $E(x)$ and $E(y)$, found above, and compute the variance of every asset,

$$\begin{aligned} Var(x) &= E(x^2) - E(x)^2 = \frac{107}{3} - 25 = \frac{32}{3} \simeq 10.66 \\ Var(y) &= E(y^2) - E(y)^2 = \frac{113}{3} - 25 = 13 \end{aligned}$$

Hence, $Var(y) > Var(x)$. Therefore, if this individual were "variance averse," he would select asset x since, given the same mean for both assets, asset x has the lowest variance.

(d) From your previous answers, comment on the validity of the following statement: "Every risk-averse individual is also variance averse".

- First, note that this individual is risk-averse since in part (a) we showed that his utility function is weakly concave. Moreover, if he were variance-averse, he would prefer the asset (or the lottery) with the lowest possible variance (asset x). However, we showed in part (b) that he chooses asset y. Hence, risk-aversion does not necessarily imply variance-aversion.

2. Consider Tony playing the following lotteries,

$$\begin{aligned} L &= (0.3, 0.5, 0.2) \\ L' &= (0.6, 0.3, 0.1) \\ L'' &= (0.25, 0.4, 0.35) \end{aligned}$$

which are the probabilities on outcomes 1, 2, and 3 respectively, subject to $\sum_{i=1}^{3} p_i = 1$.

(a) Suppose Tony weakly prefers lottery L to L' if and only if $\max_{i \in \{1,2,3\}} p_i \geq \max_{i \in \{1,2,3\}} p_i'$. Does this extreme preference for certainty violate the IA?

- First, we show that $L' \succ L$ because

$$\max_{i \in \{1,2,3\}} p_i' = 0.6 > 0.5 = \max_{i \in \{1,2,3\}} p_i$$

If the IA holds, then $L' \succ L$ must be equivalent to

$$\frac{1}{2}L' + \frac{1}{2}L'' \succ \frac{1}{2}L + \frac{1}{2}L''$$

which, in turn, is equivalent to

$$\max\left(\frac{0.6+0.25}{2},\frac{0.3+0.4}{2},\frac{0.1+0.35}{2}\right) > \max\left(\frac{0.3+0.25}{2},\frac{0.5+0.4}{2},\frac{0.2+0.35}{2}\right)$$

Simplifying, we obtain

$$\max(0.425, 0.35, 0.225) = 0.425 > 0.45 = \max(0.275, 0.45, 0.275)$$

that constitutes a contradiction. Hence, the IA does not hold.

(b) Suppose Tony weakly prefers lottery L to L' if and only if

$$\begin{aligned} &p_1 > p_1' \quad \text{or} \\ &p_1 = p_1' \quad \text{and} \quad p_2 > p_2', \quad \text{or} \\ &p_2 = p_2' \quad \text{and} \quad p_3 > p_3' \end{aligned}$$

Does this lexicographic preference violate the IA?

- First, assume that $L' \succ L$ is satisfied because

$$p_1' > p_1.$$

If the IA holds, then $L' \succ L$ is equivalent to the following convex combination of lotteries

$$\alpha L' + (1-\alpha) L'' \succ \alpha L + (1-\alpha) L''$$

where $\alpha \in [0, 1]$. This preference over compound lotteries is equivalent to

$$\alpha p_1' + (1-\alpha) p_1'' > \alpha p_1 + (1-\alpha) p_1''$$

which simplifies to $p_1' > p_1$. Since this ranking between probabilities p_1' and p_1 holds by definition, we can conclude that the IA is not violated.

3. Consider an individual with the following lexicographic preference relation over lotteries: He strictly prefers lottery L to L', $L \succ L'$, if and only if

$$\begin{aligned} &p_1 > p_1' \quad \text{or} \\ &p_1 = p_1' \quad \text{and} \quad p_2 > p_2', \quad \text{or} \\ &p_1 = p_1', \quad p_2 = p_2', \quad \text{and} \quad p_3 > p_3', \quad \text{or} \\ &\ldots \end{aligned}$$

where outcomes $i \in \{1, \ldots, N\}$ are ranked lexicographically. Illustrate with a numerical example that the lexicographic preference relation satisfies the IA.

- First, consider the following three lotteries, L, L', and L'', over three possible outcomes:

$$\begin{aligned} L &= (0.6, 0.3, 0.1) \\ L' &= (0.3, 0.5, 0.2) \\ L'' &= (0.25, 0.4, 0.35) \end{aligned}$$

  where, for a given lottery, probabilities satisfy $\sum_{i=1}^{3} p_i = 1$.

- Second, we confirm that the premise of the IA, $L \succ L'$, holds in the above lotteries because the first outcome is more likely to occur in lottery L than L', as required by the lexicographic preference relation over lotteries:

$$p_1 = 0.6 > 0.3 = p_1'$$

  By contradition, suppose that the IA does not hold. Then there must exist a probability $\pi \in [0,1]$ such that the compound lotteries $\pi L + (1-\pi) L''$ and $\pi L' + (1-\pi) L''$ satisfy

$$\pi L' + (1-\pi) L'' \succ \pi L + (1-\pi) L''.$$

  Next, we check the "compound probability" of the most preferred outcome (outcome 1). Specifically,

$$\begin{aligned} \pi p_1' + (1-\pi) p_1'' &> \pi p_1 + (1-\pi) p_1'' \\ 0.3\pi + 0.25 (1-\pi) &> 0.6\pi + 0.25 (1-\pi) \end{aligned}$$

  which simplifies to

$$0.3\pi > 0.6\pi$$

  which cannot hold for any probability $\pi \in [0,1]$. We have then reached a contradition, implying that the IA is not violated.

4. Consider an individual with the following utility function,

$$u(x) = x^{\alpha}$$

where $0 \leq \alpha \leq 1$, and $x > 0$ represents the outcome of the individual.

(a) Find the Arrow-Pratt coefficient of absolute risk aversion, $r_A(x)$.

- Differentiating the utility function with respect to x,

$$u'(x) \equiv \frac{du(x)}{dx} = \alpha x^{\alpha-1}$$
$$u''(x) \equiv \frac{d^2u(x)}{dx^2} = -\alpha(1-\alpha)x^{\alpha-2}$$

Therefore, the Arrow-Pratt coefficient of absolute risk aversion, $r_A(x)$, is

$$\begin{aligned} r_A(x) &= -\frac{u''(x)}{u'(x)} \\ &= -\frac{-\alpha(1-\alpha)x^{\alpha-2}}{\alpha x^{\alpha-1}} \\ &= \frac{1-\alpha}{x} \end{aligned}$$

(b) Suppose the individual has equal chances of obtaining x and $3x$. Find the certainty equivalent of this lottery (you should obtain an expression in terms of α and x).

- The certainty equivalent is the amount that makes this individual indifferent to the expected utility of the lottery; that is, $CE(\alpha,x)$ solves $u(CE(\alpha,x)) = E(u(x))$, which in our context entails

$$\begin{aligned} u(CE(\alpha,x)) = E(u(x)) &= \frac{1}{2}x^\alpha + \frac{1}{2}(3x)^\alpha \\ (CE(\alpha,x))^\alpha &= \frac{1+3^\alpha}{2}x^\alpha \\ \Rightarrow CE(\alpha,x) &= \left(\frac{1+3^\alpha}{2}\right)^{\frac{1}{\alpha}} x \end{aligned}$$

(c) Suppose the individual assigns probabilities w and $1-w$ to outcomes x and $3x$, respectively. What are the probabilities that make him enjoy the same utility as the utility of expected value of the lottery presented in part (b)?

- The utility of the expected value of the lottery is

$$\begin{aligned} u(E(x)) &= \left(\frac{1}{2}x + \frac{1}{2}3x\right)^\alpha \\ &= 2^\alpha x^\alpha \end{aligned}$$

- Let the individual assigns probability w to outcome x, where $w \in [0,1]$, then

$EU = u(E(x))$ entails

$$wu(x) + (1-w)\,u(3x) = u(E(x))$$
$$wx^{\alpha} + (1-w)\,3^{\alpha}x^{\alpha} = 2^{\alpha}x^{\alpha}$$

Cancelling out x^{α} on both sides (since $x \neq 0$), we obtain

$$w + (1-w)\,3^{\alpha} = 2^{\alpha}$$

Simplifying the above expression, and solving for w, yields

$$w(\alpha) = \frac{3^{\alpha} - 2^{\alpha}}{3^{\alpha} - 1}.$$

(d) Let $\alpha = \frac{1}{2}$ and $x = 1$. Evaluate the certainty equivalent in part (b) and the probability weights in part (c). Interpret your results.

- Substituting $\alpha = \frac{1}{2}$ and $x = 1$ into the certainty equivalent in part (b), we obtain

$$CE\left(\frac{1}{2}, 1\right) = \left(\frac{1+\sqrt{3}}{2}\right)^{2} = 1.866$$

  which means that the individual is willing to give up \$0.134 and receive \$1.866 as the certainty equivalent in order to avoid the risk of playing the lottery.

- Substituting $\alpha = \frac{1}{2}$ into the probability weights in part (c), we obtain

$$w\left(\frac{1}{2}\right) = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - 1} = 0.434$$