# Homework #4 (Due on September 29th, 2021)

1. Most previous exercises considered a firm producing one output with two or more types of inputs, such as labor and capital. In this exercise, we focus on the opposite setting, where a firm uses only one input $x$ to produce two outputs, $q_1, q_2 > 0$. The technology function is

$$x = A(q_1^\alpha + q_2^\beta)$$

where $\alpha, \beta > 1$. For simplicity, we assume that the firm faces perfect competition in both the input and product market, so the firm buys the input at a given price $r$ and sells the outputs at given prices $p_1$ and $p_2$.

(a) Set up the firm's profit-maximization problem. Find the firm's supply functions for good 1 and 2 (that is, its profit-maximizing outputs $q_1$ and $q_2$).

- The firm solves

$$\max_{q_1, q_2 \geq 0} \quad \pi = p_1 q_1 + p_2 q_2 - rx = \overbrace{p_1 q_1 + p_2 q_2}^{\text{Revenue}} - \overbrace{rA(q_1^\alpha + q_2^\beta)}^{\text{Costs}}$$

- Taking first-order conditions with respect to output $q_1$ and $q_2$ yields

$$\begin{aligned} \frac{\partial \pi}{\partial q_1} &= p_1 - r\alpha A q_1^{\alpha-1} = 0 \\ \frac{\partial \pi}{\partial q_2} &= p_2 - r\beta A q_2^{\beta-1} = 0 \end{aligned}$$

Solving for $q_1$ and $q_2$ in each first-order condition gives the optimal outputs for good 1 and 2.

$$\begin{aligned} q_1 &= \left(\frac{p_1}{r\alpha A}\right)^{\frac{1}{\alpha-1}} \\ q_2 &= \left(\frac{p_2}{r\beta A}\right)^{\frac{1}{\beta-1}} \end{aligned}$$

(b) Show that his production relation is strictly convex when the firm produces a positive output level of both goods, $q_1, q_2 > 0$.

- The production relation is strictly convex for $q_1, q_2 > 0$ if the principal minors of the Hessian matrix are positive within this domain. Recall that the Hessian matrix is given by

$$\text{Hess } x(q_1, q_2) = \begin{pmatrix} \frac{\partial^2 x}{\partial q_1^2} & \frac{\partial^2 x}{\partial q_1 \partial q_2} \\ \frac{\partial^2 x}{\partial q_2 \partial q_1} & \frac{\partial^2 x}{\partial q_2^2} \end{pmatrix}$$

Taking first-order conditions, we obtain

$$\begin{aligned} \frac{\partial x}{\partial q_1} &= A\alpha q_1^{\alpha-1} \\ \frac{\partial x}{\partial q_2} &= A\beta q_2^{\beta-1} \end{aligned}$$

It is straightforward to calculate the derivatives in each cell of the Hessian matrix using the above first-order conditions:

$$\begin{aligned}
\frac{\partial^2 x}{\partial q_1^2} &= \alpha(\alpha-1)Aq_1^{\alpha-2} \\
\frac{\partial^2 x}{\partial q_2^2} &= \beta(\beta-1)Aq_2^{\beta-2} \\
\frac{\partial^2 x}{\partial q_1 \partial q_2} &= \frac{\partial^2 x}{\partial q_2 \partial q_1} = 0
\end{aligned}$$

where $\frac{\partial^2 x}{\partial q_1^2} = \alpha(\alpha-1)Aq_1^{\alpha-2} > 0$ since $\alpha > 1$ by definition. Similarly, $\frac{\partial^2 x}{\partial q_2^2} = \beta(\beta-1)Aq_2^{\beta-2} > 0$ since $\beta > 1$ by definition. The principal minor is then

$$\begin{vmatrix} \alpha(\alpha-1)Aq_1^{\alpha-2} & 0 \\ 0 & \beta(\beta-1)Aq_2^{\beta-2} \end{vmatrix} = \alpha\beta A^2(\alpha-1)(\beta-1)q_1^{\alpha-2}q_2^{\beta-2} > 0$$

since the firm produces a positive output level of both goods, $q_1, q_2 > 0$, and parameter $\alpha$ and $\beta$ satisfy $\alpha, \beta > 1$ by assumption.

1. An electric car is made up of $N \geq 2$ components, $x_0, x_1, \ldots, x_N$, such as metal, wires, tires, and bearings, which must be assembled in respective proportions of $\alpha_0, \alpha_1, \ldots, \alpha_N$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$.

   (a) Write down the car manufacturer's production function. (*Hint*: It exhibits a kink at which the car is built up in the exact proportion of every component.)

   - Since the car manufacturer uses proportion $\alpha_i$ of component $x_i$ to build every car, its production function exhibits a kink at $(\alpha_0 x_0, \alpha_1 x_1, \ldots, \alpha_N x_N)$, so that the ray from the origin to the kink has a gradient vector of

     $$\nabla f = \left(\frac{x_1}{x_0}, \ldots, \frac{x_N}{x_0}\right) = \left(\frac{\alpha_1}{\alpha_0}, \ldots, \frac{\alpha_N}{\alpha_0}\right)$$

     where we take component $x_0$ as the basis. Then, for every component $i = \{1, \ldots, N\}$, we have a proportion ratio

     $$\frac{x_i}{x_0} = \frac{\alpha_i}{\alpha_0}$$

   - Simplifying, the production function of the car manufacturer becomes

     $$f(x_0, x_1, \ldots, x_N) = \min\left\{x_0, \frac{\alpha_0}{\alpha_1}x_1, \ldots, \frac{\alpha_0}{\alpha_N}x_N\right\}$$

     so that the car manufacturer regards every component to be complementary to one another. Intuitively, using more of one component without a commensurate increase of the other components does not increase the units of cars produced.

(b) Let $p_i$ be the price of component $i$, and let the car manufacturer be a price taker. Assume that the total cost that the car manufacturer seeks to achieve is $TC$, yielding an average cost $c = \frac{TC}{q}$ in building every electric car. Derive the demand function for every component. For simplicity, you can normalize the price of component $x_0$ to $p_0 = 1$.

- The car manufacturer chooses units of component $x_i$ to solve

$$\max_{\{x_i\}_{i=0}^{N}} \min\left\{x_0, \frac{\alpha_0}{\alpha_1}x_1, \ldots, \frac{\alpha_0}{\alpha_N}x_N\right\}$$
$$\text{subject to} \quad x_0 + p_1x_1 + \ldots + p_Nx_N = c$$

  Since the car manufacturer assembles the components in designated proportions (that is, at the kink), substituting $x_i = \frac{\alpha_i}{\alpha_0}x_0$ into the budget constraint, we find

$$x_0 + p_1\underbrace{\frac{\alpha_1}{\alpha_0}x_0}_{x_1} + \ldots + p_N\underbrace{\frac{\alpha_N}{\alpha_0}x_0}_{x_N} = c$$

  which we rearrange to yield the demand function for component $x_0$, as follows

$$q_0(\alpha, p, c) = \frac{\alpha_0 c}{\alpha_0 + \alpha_1p_1 + \ldots + \alpha_Np_N}$$

- Inserting this demand function into $x_i = \frac{\alpha_i}{\alpha_0}x_0$, the demand for component $x_i$ is

$$q_i(\alpha, p, c) = \frac{\alpha_i}{\alpha_0}\underbrace{\frac{\alpha c}{\alpha_0 + \alpha_1p_1 + \ldots + \alpha_Np_N}}_{x_0}$$
$$= \frac{\alpha_i c}{\alpha_0 + \alpha_1p_1 + \ldots + \alpha_Np_N}$$

  The demand of every component is increasing in the average cost that the car manufacturer seeks to incur, $c$, but decreasing in the price of every component (including its own), since it assembles the components in fixed proportions. Formally, $q_i(\alpha, p, c)$ is decreasing in $p_i$ and in $p_j$ for every component $j \neq i$.

(c) Find the car manufacturer's output function.

- Substituting each component's demand into the production function, the output function is

$$f(\alpha, p, c) = \min\left\{\underbrace{\frac{\alpha_0 c}{\alpha_0 + \alpha_1p_1 + \ldots + \alpha_Np_N}}_{x_0},\right.$$
$$\left.\frac{\alpha_0}{\alpha_1}\underbrace{\frac{\alpha_1 c}{\alpha_0 + \alpha_1p_1 + \ldots + \alpha_Np_N}}_{x_1}, \ldots, \frac{\alpha_0}{\alpha_N}\underbrace{\frac{\alpha_N c}{\alpha_0 + \alpha_1p_1 + \ldots + \alpha_Np_N}}_{x_N}\right\}$$
$$= \frac{\alpha_0 c}{\alpha_0 + \alpha_1p_1 + \ldots + \alpha_Np_N}$$

which is, as expected, increasing in $c$, but decreasing in every component's price. Intuitively, when the price of any component increases, the car manufacturer decreases its demand of both this component and all other components, ultimately being able to produce fewer cars.

2. Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce one type of output. The price of every unit of input is $w = 8$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem, and solve for its unconditional factor demand $z(8, p)$.

   - The firm chooses the units of input $z$ to solve

   $$\max_{z \geq 0} p\sqrt{z} - 8z$$

   where the first term indicates total revenue, whereas the second reflects total costs. Taking first-order condition with respect to $z$, we obtain

   $$p\frac{1}{2}z^{-1/2} - 8 \leq 0.$$

   In the case of interior solutions, we can solve for $z$ to find the unconditional factor demand

   $$z(8, p) = \frac{p^2}{256}.$$

   Hence, total output is $q = \sqrt{\frac{p^2}{256}} = \frac{p}{16}$ units.

   (b) Evaluate the profit function at the unconditional factor demand $z(8, p)$. Test for convexity of the profit function in output price $p$.

   - Inserting $z(8, p) = \frac{p^2}{256}$ into the firm's objective function, we obtain

   $$\pi(p) = p\sqrt{z(8, p)} - 8z(8, p) = \frac{1}{32}p\,(2p - p) = \frac{p^2}{32},$$

   which is convex in output price $p$.

   (c) Let us now illustrate convexity in output prices by using an alternative approach: (1) evaluate the profit function you found in part (b) at prices $p = 6$, and at $p = 12$. Then, find their convex combination $\alpha\pi(6) + (1 - \alpha)\,\pi(12)$ where $\alpha \in [0, 1]$; (2) evaluate the profit function at the convex combination of the above output prices, that is, $\pi(\alpha 6 + (1 - \alpha)\,12)$. Last, show that the profit function you found in step (1) lies weakly above that found in step (2) for all values of $\alpha$, that is,

   $$\alpha\pi(6) + (1 - \alpha)\,\pi(12) \geq \pi(\alpha 6 + (1 - \alpha)\,12).$$

   - Evaluating the output function at those two output prices, we obtain $\pi(6) = \frac{9}{8}$ and $\pi(12) = \frac{9}{2}$. Hence, their convex combination is

   $$\alpha\pi(6) + (1 - \alpha)\,\pi(12) = \alpha\frac{9}{8} + (1 - \alpha)\,\frac{9}{2} = \frac{9}{8}(4 - 3\alpha).$$

If, instead, we evaluate the profit function at an output price $p = \alpha 6 + (1-\alpha)\,12$, we obtain

$$\pi\left(\alpha 6 + (1-\alpha)\,12\right) = \frac{9}{8}(4 - 4\alpha + \alpha^2)$$

Substracting $[\alpha\pi(6) + (1-\alpha)\,\pi(12)] - \pi(\alpha 6 + (1-\alpha)\,12)$, we find

$$\frac{9}{8}(4 - 3\alpha) - \frac{9}{8}(4 - 4\alpha + \alpha^2) = \frac{9}{8}\alpha\,(1-\alpha) > 0.$$

which is positive since $\alpha \in [0, 1]$.