# Recitation (09/24/2021)

1. Consider the following profit function that has been obtained from a technology that uses a single input, $z$:

$$\pi(p, w) = p^2 w^\alpha$$

where $p$ is the output price, $w$ is the input price and $\alpha$ is a parameter value.

(a) Check if the profit function satisfies homogeneity of degree one jointly in both $p$ and $w$. In particular, determine for which values of $\alpha$ this property is satisfied.

- The profit function is homogeneous of degree one if

$$\pi(\theta p, \theta w) = \theta \pi(p, w)$$

In this case we have that the left-hand term becomes

$$\pi(\theta p, \theta w) = (\theta p)^2 (\theta w)^\alpha = \theta^{2+\alpha} p^2 w^\alpha \tag{3}$$

and, on the other hand, the right-hand term is

$$\theta \pi(p, w) = \theta p^2 w^\alpha \tag{4}$$

since, by homogeneity of degree one, expressions (3) and (4) must coincide. Then,

$$\theta^{2+\alpha} p^2 w^\alpha = \theta p^2 w^\alpha$$

which implies that $2 + \alpha = 1$. That is, we need $\alpha = -1$. As a consequence, the profit function that we obtain is

$$\pi(p, w) = \frac{p^2}{w}$$

(b) Assuming the value of $\alpha$ for which the profit function satisfies homogeneity of degree one, check if the profit function $\pi(p, w)$ satisfies the following properties: (1) non-decreasing in output price $p$, (2) non-increasing in input prices $w$, and (3) convex in prices $p$ and $w$.

- *Non-decreasing in the output price, $p$:* Increasing output prices yields a weakly higher profit level since

$$\frac{\partial \pi(p, w)}{\partial p} = \frac{2p}{w} \geq 0$$

- *Non-increasing in the factor prices, w:* Increasing all input prices weakly reduces profits since

$$\frac{\partial \pi(p, w)}{\partial w} = -\frac{p^2}{w^2} \leq 0$$

- *Convex in prices* (factor prices and output prices):

$$\begin{vmatrix} \frac{\partial^2 \pi(p,w)}{\partial p^2} & \frac{\partial^2 \pi(p,w)}{\partial p \partial w} \\ \frac{\partial^2 \pi(p,w)}{\partial w \partial p} & \frac{\partial^2 \pi(p,w)}{\partial w^2} \end{vmatrix} = \begin{vmatrix} \frac{2}{w} & -\frac{2p}{w^2} \\ -\frac{2p}{w^2} & \frac{2p^2}{w^3} \end{vmatrix}$$

  In particular, the Hessian is a positive semi-definite matrix, since

$$\frac{2}{w}\frac{2p^2}{w^3} - \left(-\frac{2p}{w^2}\right)\left(-\frac{2p}{w^2}\right) = \frac{4p^2}{w^4} - \frac{4p^2}{w^4} = 0$$

  implying that the profit function $\pi(p, w)$ is convex.

(c) Calculate the supply function of the firm, $q(p, w)$, and its demand for inputs, $z(p, w)$.

- Using Hotelling's Lemma we can find the supply function, by differentiating the profit function with respect to $p$, as follows

$$q(p,\, w) = \frac{\partial \pi(p, w)}{\partial p} = \frac{2p}{w}$$

  and the conditional factor demand correspondence can also be found by differentiating the profit function with respect to $w$, as follows

$$z(p,\, w) = -\frac{\partial \pi(p, w)}{\partial w} = \frac{p^2}{w^2}$$

- Note that both the supply function, $q(p,\, w)$, and the input demand function, $z(p,\, w)$, are increasing in output prices $p$ (more attractive sales) but decreasing in input prices $w$ (i.e., more costly resources).

2. Suppose that a firm owns two plants, each producing the same good. Every plant $j$'s average cost is given by

$$AC_j(q_j) = \alpha + \beta_j q_j \;\text{ for }\; q_j \geq 0, \text{ where } j = \{1, 2\}$$

where coefficient $\beta_j$ may differ from plant to plant, i.e., if $\beta_1 > \beta_2$ plant 2 is more efficient than plant 1 since its average costs increase less rapidly in output. Assume that you are asked to determine the cost-minimizing distribution of aggregate output $q = q_1 + q_2$, among the two plants (i.e., for a given aggregate output $q$, how much $q_1$

to produce in plant 1 and how much $q_2$ to produce in plant 2.) For simplicity, consider that aggregate output $q$ satisfies $q < \frac{\alpha}{\max\limits_j |\beta_j|}$. (You will be using this condition in part b.)

(a) If $\beta_j > 0$ for every plant $j$, how should output be located among the two plants?

- The cost-minimization problem in which we find the optimal combination of output $q_1$ and $q_2$ that minimizes the total cost of production across plants is

$$\min_{q_1, q_2} \; TC_1(q_1) + TC_2(q_2)$$

$$\text{subject to} \;\; q_1 + q_2 = q$$

or equivalently, the profit maximization problem in which firms choose the optimal combination of output $q_1$ and $q_2$ that maximizes the total profits across all plants is

$$\max_{q_1, q_2} \; \underbrace{pq_1 - TC_1(q_1)}_{\pi_1} + \underbrace{pq_2 - TC_2(q_2)}_{\pi_2}$$

$$\text{subject to} \;\; q_1 + q_2 = q$$

- If the average cost is $AC_j(q_j) = \alpha + \beta_j q_j$ then the total cost is $TC_j(q_j) = (\alpha + \beta_j q_j)q_j$. Thus, we can rewrite the above PMP as:

$$\max_{q_1, q_2} \; pq_1 - (\alpha + \beta_1 q_1)q_1 + pq_2 - (\alpha + \beta_2 q_2)q_2$$

$$\text{subject to} \;\; q_1 + q_2 = q$$

Taking first order conditions with respect to $q_1$ and $q_2$ yields

$$\frac{\partial(\pi_1 + \pi_2)}{\partial q_1} = p - \alpha - 2\beta_1 q_1 = \lambda$$

$$\frac{\partial(\pi_1 + \pi_2)}{\partial q_2} = p - \alpha - 2\beta_2 q_2 = \lambda$$

$$\frac{\partial(\pi_1 + \pi_2)}{\partial \lambda} = q_1 + q_2 = q$$

Using the first two order conditions, we obtain

$$p - \alpha - 2\beta_1 q_1 = p - \alpha - 2\beta_2 q_2$$

and rearranging, $q_2 = \frac{\beta_1}{\beta_2} q_1$. Replacing this expression into the constraint

$q_1 + q_2 = q$ yields

$$q_1 + \underbrace{\frac{\beta_1}{\beta_2} q_1}_{q_2} = q$$

and solving for $q_1$ entails the cost-minimizing production in plant 1,

$$q_1 \left(1 + \frac{\beta_1}{\beta_2}\right) = q, \quad \text{thus} \quad q_1 = \frac{\beta_2}{\beta_1 + \beta_2} q,$$

and operating similarly for $q_2$, we find

$$q_2 = \frac{\beta_1}{\beta_1 + \beta_2} q$$

- *Extension:* Note that, generally for $J$ plants, the average cost of plant $j$ is $AC_j(q_j) = \alpha + \beta_j q_j$ implying that the total cost must be $TC_j(q_j) = (\alpha + \beta_j q_j) q_j$. Therefore, plant $j$'s marginal cost is $MC_j(q_j) = \alpha + 2\beta_j q_j$. Since $\beta_j > 0$ for every $j$, the first order necessary and sufficient conditions for cost minimization are: (1) that firms' marginal costs coincide (otherwise, we would still have incentives to distribute a larger production to those firms with the lowest marginal cost)

$$MC_j(q_j) = MC_{j'}(q_{j'}) \text{ for any two plants } j \text{ and } j'$$

and; (2) that the aggregate output constraint holds

$$q_1 + q_2 + ...q_J = q.$$

From these conditions we obtain

$$q_j = \frac{\frac{q}{\beta_j}}{\sum_h \frac{1}{\beta_h}}.$$

which coincides with our results for $N = 2$ plants,

$$q_1 = \frac{\frac{q}{\beta_1}}{\frac{1}{\beta_1} + \frac{1}{\beta_2}} = \frac{\beta_2}{\beta_1 + \beta_2} q.$$

Figure 2 depicts the average and marginal cost curves for two plants satisfying $\beta_2 > \beta_1$. In particular, the firm manager chooses, for a given aggregate output $q = q_1 + q_2$, the individual output levels $q_1$ and $q_2$ that equate the marginal costs across both plants (see vertical axis).

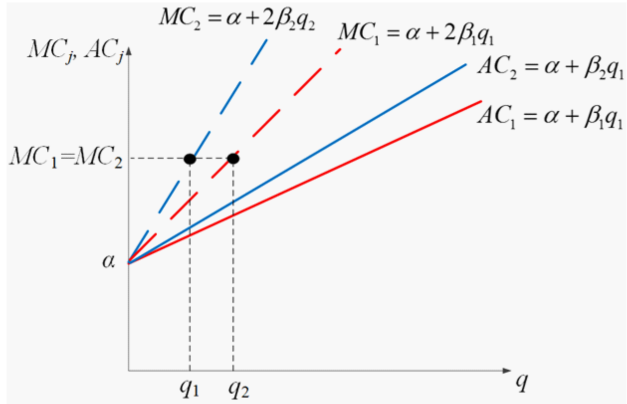


Figure 2. $\beta_j > 0$ for every firm.

(b) If $\beta_j < 0$ for every plant $j$, how should output be located among the two plants?

- First, note that $\beta_j < 0$ implies that the average cost $AC_j(q_j) = \alpha + \beta_j q_j$ is decreasing in output. Hence, it is cost-minimizing to concentrate all production on the plant with the smallest $\beta_j < 0$ (the most negative $\beta_j$) because average costs (and total costs) are minimized by doing so.
- Figure 3 depicts a firm in which both plants exhibit decreasing average costs, but $\beta_2 < \beta_1 < 0$, implying that it is beneficial for the firm to concentrate all output in plant 2. In addition, note that the average cost in plant 1 is positive for all $q_1$ as long as $\alpha - \beta_1 q_1 > 0$, or $q_1 < \frac{\alpha}{\beta_1}$, where $\frac{\alpha}{\beta_1}$ represents the horizontal intercept of $AC_1$ in the figure. Similarly for firm 2, where $AC_2 > 0$ for all $q_2$ as long as $q_2 < \frac{\alpha}{\beta_2}$, where $\frac{\alpha}{\beta_2}$ represents the horizontal intercept of $AC_2$. Hence, the original condition $q < \frac{\alpha}{\max_j |\beta_j|}$ is equivalent to $q < \min_j \frac{\alpha}{|\beta_j|}$, graphically implying that the aggregate output $q$ lies to the left-hand side to

the smallest horizontal intercept.

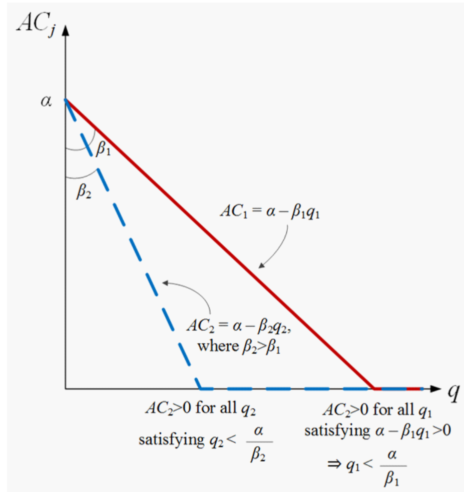


Figure 3. $\beta_j < 0$ for every firm.

(c) If $\beta_j > 0$ for some plants and $\beta_i < 0$ for others?

- Similarly as in part (b), the firm now faces some plants with increasing average costs (those with $\beta_j > 0$) and some plants with decreasing average costs (those with $\beta_j < 0$). Hence, it is cost-minimizing to concentrate all production on the plant/s with the smallest $\beta_j < 0$, since it benefits from the most rapidly decreasing average costs. Figure 4 depicts a firm with plant 1 (2) having increasing (decreasing, respectively) average costs.

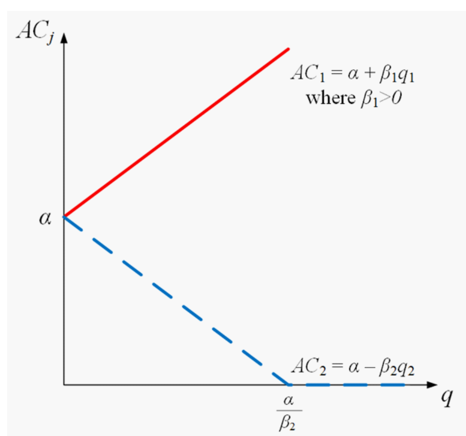


Figure 4. $\beta_1 > 0$ and $\beta_2 < 0$.