Homework #4 (Due on October 4th, 2021)

1. Most previous exercises considered a firm producing one output with two or more types of inputs, such as labor and capital. In this exercise, we focus on the opposite setting, where a firm uses only one input x to produce two outputs, $q_1, q_2 > 0$. The technology function is

$$x = A(q_1^\alpha + q_2^\beta)$$

where $\alpha, \beta > 1$. For simplicity, we assume that the firm faces perfect competition in both the input and product market, so the firm buys the input at a given price r and sells the outputs at given prices p_1 and p_2.

   (a) Set up the firm's profit-maximization problem. Find the firm's supply functions for good 1 and 2 (that is, its profit-maximizing outputs q_1 and q_2).

   (b) Show that his production relation is strictly convex when the firm produces a positive output level of both goods, $q_1, q_2 > 0$.

2. An electric car is made up of $N \geq 2$ components, $x_0, x_1, \ldots, x_N$, such as metal, wires, tires, and bearings, which must be assembled in respective proportions of $\alpha_0, \alpha_1, \ldots, \alpha_N$, where $\alpha_i \geq 0$ and $\sum_{i=1}^N \alpha_i = 1$.

   (a) Write down the car manufacturer's production function. (*Hint*: It exhibits a kink at which the car is built up in the exact proportion of every component.)

   (b) Let p_i be the price of component i, and let the car manufacturer be a price taker. Assume that the total cost that the car manufacturer seeks to achieve is TC, yielding an average cost $c = \frac{TC}{q}$ in building every electric car. Derive the demand function for every component. For simplicity, you can normalize the price of component x_0 to $p_0 = 1$.

   (c) Find the car manufacturer's output function.

3. Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce one type of output. The price of every unit of input is $w = 8$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem, and solve for its unconditional factor demand $z(8, p)$.

   (b) Evaluate the profit function at the unconditional factor demand $z(8, p)$. Test for convexity of the profit function in output price p.

   (c) Let us now illustrate convexity in output prices by using an alternative approach: (1) evaluate the profit function you found in part (b) at prices $p = 6$, and at $p = 12$. Then, find their convex combination $\alpha\pi(6) + (1 - \alpha)\,\pi(12)$ where $\alpha \in [0, 1]$; (2) evaluate the profit function at the convex combination of the above output prices, that is, $\pi(\alpha 6 + (1 - \alpha)\,12)$. Last, show that the profit function you found in step (1) lies weakly above that found in step (2) for all values of α, that is,

$$\alpha\pi(6) + (1 - \alpha)\,\pi(12) \geq \pi(\alpha 6 + (1 - \alpha)\,12).$$