Homework #3 (Due on September 21st, 2021)

1. After many years of hard work, your high school teacher has retired and she receives a fixed income $w > 0$ which does not adjust to inflation. Her expenditure function is

$$e(p_1, p_2, u) = (p_1 + p_2)^2 u,$$

where $p_1, p_2 > 0$ denote initial prices. Suppose that prices of goods 1 and 2 increase to $\tilde{p}_1$ and $\tilde{p}_2$, respectively.

   (a) You would like to give her a monetary gift so that she will not be affected by the above price increase. How much money should you give her? That is, find his compensating variation (CV).

   - From duality, we know that $e(p_1, p_2, u) = w$, which helps us rewrite the above equation as $w = (p_1 + p_2)^2 u$. Therefore, at the initial price vector, the maximum utility the retiree can obtain is $u = \frac{w}{(p_1+p_2)^2}$. In order to ensure that she is not worse off after the price increase, we need to give her an amount of money that covers the difference between the new expense and the old expense, that is,

$$\begin{aligned} CV &= \tilde{w} - w \\ &= e(\tilde{p}_1, \tilde{p}_2, u) - w \\ &= \underbrace{(\tilde{p}_1 + \tilde{p}_2)^2 u}_{e(\tilde{p}_1, \tilde{p}_2, u)} - w \\ &= (\tilde{p}_1 + \tilde{p}_2)^2 \underbrace{\frac{w}{(p_1 + p_2)^2}}_{u} . - w \\ &= w \left(\frac{(\tilde{p}_1 + \tilde{p}_2)^2}{(p_1 + p_2)^2} - 1 \right) \end{aligned}$$

   (b) Now find her equivalent variation (EV) from the price change.

- The equivalent variation is

$$
\begin{aligned}
EV &= e(p_1, p_2, \widetilde{u}) - e(\tilde{p}_1, \tilde{p}_2, \widetilde{u}) \\
&= (p_1 + p_2)^2 \widetilde{u} - (\tilde{p}_1 + \tilde{p}_2)^2 \widetilde{u} \\
&= (p_1 + p_2)^2 \underbrace{\frac{w}{(\tilde{p}_1 + \tilde{p}_2)^2}}_{\widetilde{u}} - (\tilde{p}_1 + \tilde{p}_2)^2 \underbrace{\frac{w}{(\tilde{p}_1 + \tilde{p}_2)^2}}_{\widetilde{u}} \\
&= w \left(\frac{(p_1 + p_2)^2}{(\tilde{p}_1 + \tilde{p}_2)^2} - 1 \right)
\end{aligned}
$$

(c) Which is larger in this case, CV or EV?

- In this case, since $\tilde{p}_1 + \tilde{p}_2 > p_1 + p_2$, we obtain that the compensating variation is positive $CV > 0$, but the equivalent variation is negative $EV < 0$, entailing that $CV > 0 > EV$.

(d) Find his Walrasian demand for each good.

- *Good 1.* From duality, we know that $e(p_1, p_2, u) = w$, which yields $w = (p_1 + p_2)^2 u$. Solving for u, we find the indirect utility function

$$v(p_1, p_2) = (p_1 + p_2)^{-2} w$$

Then, we can insert this indirect utility function into Roy's identity, as follows, which yields the Walrasian demand for good 1:

$$x_1(p_1, p_2, w) = -\frac{\frac{\partial v(p_1,p_2)}{\partial p_1}}{\frac{\partial v(p_1,p_2)}{\partial w}} = -\frac{-2w(p_1 + p_2)^{-3}}{(p_1 + p_2)^{-2}} = 2w(p_1 + p_2)^{-1}.$$

- *Good 2.* Following a similar approach, we can find the Walrasian demand for good 2:

$$x_2(p_1, p_2, w) = -\frac{\frac{\partial v(p_1,p_2)}{\partial p_2}}{\frac{\partial v(p_1,p_2)}{\partial w}} = -\frac{-2w(p_1 + p_2)^{-3}}{(p_1 + p_2)^{-2}} = 2w(p_1 + p_2)^{-1}$$

implying that the consumer purchases the same amount of both goods, $x_1(p_1, p_2, w) = x_2(p_1, p_2, w)$.

(e) Find her utility function. What is this type of utility function called?

- The utility function must yield the same consumption of both goods but cannot be a Leontieff type of utility function $u(x_1, x_2) = A \min\{x_1, x_2\}$, where $A > 0$ as, otherwise, the expenditure function would be $e = (p_1 + p_2)u$. The expenditure function we started with in this exercise was quite odd, because it's not well behaved (in the sense of satisfying homogeneity of degree one) and because there are no utility functions (even variations of the standard ones) that could generate such expenditure function.

2. Show that the compensating and the equivalent variation coincide when the utility function is quasilinear with respect to the first good (and we fix $p_1 = 1$). [*Hint*: Use the definitions of the compensating and equivalent variations in terms of the expenditure function (not the hicksian demand). In addition, recall that if $u(x)$ is quasilinear with respect to good 1, then we can express it as

$$u(x) = x_1 + \phi(x_{-1}),$$

where x_{-1} represents all the remaining goods, $l = 2, 3, ..., L$.]

- From the definition of the compensating and the equivalent variation, we know that

$$\begin{aligned} CV\left(p^0, p^1, w\right) &= e\left(p^1, u^1\right) - e\left(p^1, u^0\right) \\ EV\left(p^0, p^1, w\right) &= e\left(p^0, u^1\right) - e\left(p^0, u^0\right) \end{aligned}$$

Moreover, we know that if $u(x)$ is quasilinear with respect to good 1, then we can express it as

$$u(x) = x_1 + \phi(x_{-1}) \Longleftrightarrow x_1 = u(x) - \phi(x_{-1})$$

where x_{-1} represents all the remaining goods, $l = 2, 3, ..., L$. Therefore, the expenditure function becomes

$$e(p, u) = \sum_{i=1}^{L} p_i x_i = \underbrace{p_1}_{\$1} x_1 + \underbrace{\sum_{k=2}^{L} p_k x_k}_{p_{-1} \cdot x_{-1}} = x_1 + p_{-1} \cdot x_{-1}$$

where we use the fact that $p_1 = 1$. Substituting x_1 from the above expression of the quasilinear utility function, we have

$$e(p, u) = \underbrace{u(x) - \phi(x_{-1}(p_{-1}))}_{x_1} + p_{-1} \cdot x_{-1}$$

Using this expression for the expenditure function into the above definition of the compensating variation, we obtain

$$\begin{aligned} CV\left(p^0, p^1, w\right) &= e\left(p^1, u^1\right) - e\left(p^1, u^0\right) \\ &= \left[u^1(x) - \phi\left(x_{-1}(p_{-1})\right) + p_{-1} \cdot x_{-1}\right] \\ &\quad - \left[u^0(x) - \phi\left(x_{-1}(p_{-1})\right) + p_{-1} \cdot x_{-1}\right] \\ &= u^1(x) - u^0(x) \end{aligned}$$

And similarly for the definition of the equivalent variation,

$$\begin{aligned} EV\left(p^0, p^1, w\right) &= e\left(p^0, u^1\right) - e\left(p^0, u^0\right) \\ &= \left[u^1(x) - \phi\left(x_{-1}(p_{-1})\right) + p_{-1} \cdot x_{-1}\right] \\ &\quad - \left[u^0(x) - \phi\left(x_{-1}(p_{-1})\right) + p_{-1} \cdot x_{-1}\right] \\ &= u^1(x) - u^0(x) \end{aligned}$$

Therefore, for quasilinear utility functions, the compensating and the equivalent variation give us the *same* measures of the monetary value, $u^1(x) - u^0(x)$, that a consumer would assign to a reduction (or increase) in the price of a good. Recall that this is because quasilinear utility functions do not generate income effects, and when income effects are absent, the equivalent variation coincides with the compensating variation, and they both coincide with the change in consumer surplus.

3. Consider a consumer with the following expenditure function

$$e(p, u^0) = g(p) + \left[u^0 \times f(p)\right]$$

where functions $g(p)$ and $f(p)$ depend on the price vector p alone. Show that a 1% increase in wealth leads to exactly a 1% increase in consumption (i.e., the income elasticity, $\varepsilon_{x_i,w}$) converges to one when the consumer's wealth level tends to infinity, i.e., $\lim_{w\to\infty} \varepsilon_{x_i,w} = 1$.

- The income-elasticity of good i is given by

$$\varepsilon_{x_i,w} = \frac{\partial x_i}{\partial w}\frac{w}{x_i}.$$

We hence need to first find the Walrasian demand associated to this expenditure function. In order to find it, we need to do it in two steps: first, use the expenditure function $e(p, u^0)$ to obtain the indirect utility function $v(p, w)$; and second, use the indirect utility function $v(p, w)$ to obtain the Walrasian demand of good i, $x_i(p, w)$.

- **First step.** *From* $e(p, u^0)$ *to* $v(p, w)$. Recall that, by the duality theorem, the minimal expenditure needed in order to reach the utility level that the consumer attains after solving his UMP, $v(p, w)$, is $e(p, v(p, w)) = w$, which in this exercise entails

$$g(p) + [v(p, w) \times f(p)] = w,$$

and solving for $v(p, w)$, we find that the indirect utility function is

$$v(p, w) = \frac{w - g(p)}{f(p)}.$$

- **Second step.** *From* $v(p, w)$ *to* $x_i(p, w)$. Once we found the consumer's indirect utility function, $v(p, w)$, we can now use Roy's identity to obtain the Walrasian demand,

$$\begin{aligned} x_i(p, w) &= -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\frac{-g_i(p)f(p) - f_i(p)[w - g(p)]}{f(p)^2}}{\frac{1}{f(p)}} = \\ &= g_i(p) + \frac{f_i(p)}{f(p)}[w - g(p)] \end{aligned}$$

where, for compactness, we denote $g_i(p) \equiv \frac{\partial g(p)}{\partial p_i}$ and similarly $f_i(p) \equiv \frac{\partial f(p)}{\partial p_i}$.

- After finding the Walrasian demand $x_i(p, w)$, we can identify the income elasticity of good i, as follows

$$\begin{aligned} \varepsilon_{x_i,w} &= \frac{\partial x_i}{\partial w}\frac{w}{x_i} = \frac{f_i(p)}{f(p)} \frac{w}{g_i(p) + \frac{f_i(p)}{f(p)}[w - g(p)]} = \\ &= \frac{f_i(p)w}{g_i(p)f(p) + f_i(p)w - f_i(p)g(p)} \end{aligned}$$

and taking the limit of this ratio when $w \to \infty$, we obtain that $\lim_{w \to \infty} \varepsilon_{x_i,w} = 1$.

4. Consider an Italian household that is seen to purchase quantities of just two goods, spaghetti and cheese. Denote quantities of spaghetti by x and quantities of cheese by y. The household comprises two individuals: Alessandro, whose preference relation can be represented by the utility function $u_A(x, y) = x$ (he is a spaghetti lover) and Beatrice, whose preference relation can be represented by the utility function $u_B(x, y) = y$ (she is a cheese lover).

   (a) Find the Walrasian demand functions for both Alessandro and Beatrice.

- Alessandro's utility maximization problem can be represented as follows

$$\max_{x,y} u_A(x, y) = x$$

$$\text{subject to} \;\; p_x x + p_y y \leq w_A$$

Alessandro will spend all wealth on spaghetti, good x, and buy no cheese (since his utility function is independent on good y). Thus, Alessandro's demand is

$$x_A\left(p_x, p_y, w_A\right) = \frac{w_A}{p_x} \quad \text{and} \quad y_A\left(p_x, p_y, w_A\right) = 0$$

- In contrast, Beatrice maximizes her utility by spending all her wealth on cheese (since good x does not enter in her utility function). Hence, Beatrice's demand is

$$x_B\left(p_x, p_y, w_B\right) = 0 \quad \text{and} \quad y_B\left(p_x, p_y, w_B\right) = \frac{w_B}{p_y}$$

(b) Consider that, upon receiving a household wealth level of w, Alessandro and Beatrice agree on evenly dividing the wealth between them (otherwise, a major Italian fight will ensue!). Suppose that you observe the aggregate demands of this household and you interpret it as if it originated from just a single representative consumer. Find the demands of the representative consumer.

- If a positive representative consumer exists, its demand will equal the aggregate demand of the household. That is:

$$x(p_x, p_y, w) = x_A\left(p_x, p_y, \frac{w}{2}\right) + x_B\left(p_x, p_y, \frac{w}{2}\right) = \frac{w}{2p_x} + 0$$

$$y(p_x, p_y, w) = y_A\left(p_x, p_y, \frac{w}{2}\right) + y_B\left(p_x, p_y, \frac{w}{2}\right) = 0 + \frac{w}{2p_y}$$

(c) Assume that the Italian government considers introducing a tax on spaghetti in order to maintain the luxury parties some of their politicians are famous for, and they hire you to evaluate the welfare effects associated to this tax. Recall that the equivalent variation of a change in prices, from p^0 to p^1 is defined as:

$$EV = e(p^0, v(p^1, w^0)) - e(p^0, v(p^0, w^0))$$

If the change in prices is caused by the imposition of a commodity tax on spaghetti, then the deadweight loss (DWL), or excess burden of the tax, is given by:

$$DWL = EV - \sum_{l=1}^{L} t_l x_l(p^1, w^0)$$

where t_l represents the price change, i.e., $t_l = p_l^1 - p_l^0$, since the new price captures the initial price plus the tax, $p_l^1 = p_l^0 + t_l$. Briefly explain why this measure may be viewed as a deadweight loss for the Italian society.

- First, note that $\sum_{l=1}^{L} t_l x_l(p^1, w^0)$ is the tax revenue raised by the tax. EV is the change in wealth that would leave the representative consumer as well off as he was before the price change induced by the tax. Alternatively, EV represents how much tax revenue the government could have raised from the individual through a lump-sum (i.e., non-distortionary) tax, that would leave him as well off as under the tax on spaghetti.
- Hence, the difference between how much the Italian government could have raised through a lump-sum tax and how much it actually raised through distortionary taxes measures the excess burden associated with the distortionary commodity tax, relative to the non-distortionary lump-sum tax on wealth. This excess burden measures a welfare loss for society, since it identifies a welfare level that is not redistributed from one agent to another in this economy, but a net loss.

(d) Suppose that this Italian household initially faces prices $p^0 = (1, 2)$ and has wealth $w^0 = 300$. In this context, a specific tax of 2 is imposed on spaghetti (i.e., good x) that leads to its price rising to $p_x = 3$ (with the price of cheese, i.e., good y, and the households' wealth both remaining unchanged). Calculate the DWL under the assumption that the household demands originate from one representative consumer.

- From part (b) of the exercise we have that the demand of the representative consumer is

$$x(p_x, p_y, w) = \frac{w}{2p_x} \text{ and } y(p_x, p_y, w) = \frac{w}{2p_y}$$

Importantly, these Walrasian demands emerge from an individual with Cobb-Douglas utility function $u(x, y) = xy$ (or equivalently, $\ln x + \ln y$). Given this utility function, his indirect utility function $v(p_x, p_y, w)$ becomes

$$v(p_x, p_y, w) = x(p_x, p_y, w) * y(p_x, p_y, w)$$

$$= \frac{w^2}{4p_x p_y}$$

In order to obtain the DWL that emerges from setting a commodity tax on the purchases of this representative consumer, we just need to find the expenditure function that arise from the above indirect utility function and, afterwards, use such expenditure to measure the EV from the imposition of the tax.

- To find the expenditure function, we can first use the identity:

$$v(p_x, p_y, e(p_x, p_y, u)) \equiv u \Leftrightarrow \frac{[e(p_x, p_y, u)]^2}{4p_x p_y} \equiv u$$

and solving for $e(p_x, p_y, u)$ yields

$$e(p_x, p_y, u) = 2\sqrt{p_x p_y u}$$

Hence, the equivalent variation from the introduction of the commodity tax is

$$EV = e(p^0, v(p^0, w)) - e(p^0, v(p^1, w))$$

using the property that $e(p^0, v(p^0, w)) \equiv w$, the EV becomes

$$EV = w - 2\sqrt{p_x^0 p_y^0 v(p_x^1, p_y^1, w)}$$

Plugging $v(p_x, p_y, w) = \frac{w^2}{4p_x p_y}$ and rearranging, the EV simplifies to

$$EV = 300 - 2\sqrt{2\frac{(300)^2}{4 * 2 * 3}} = (3 - \sqrt{3}) \times 100 \simeq 126.79$$

On the other hand, the tax revenue collected is

$$\sum_{l=1}^{L} t_l x_l(p^1, w) = 2 \times x(3, 2, 300) = 2 \times 50 = 100$$

Therefore, the DWL from imposing the tax is

$$DWL = (3 - \sqrt{3}) \times 100 - 100 \approx 26.79$$

(e) Without assuming the existence of a representative consumer, use the individuals' indirect utility functions derived in part (a) to calculate the two individual DWLs. Does its sum coincide with the deadweight loss you found in part (d) where we assumed the existence of a representative consumer?

- As we did in the previous part of the exercise, in order to find the deadweight loss of the tax, we first have to identify every individual's expenditure function which, afterwards, will help us find the equivalent variation from introducing the tax.
- Let us first examine Alessandro (the spaghetti lover). From part (a) of the exercise, we found that Alessandro's indirect utility function is:

$$v_A(p_x, p_y, w_A) = x_A(p_x, p_y, w_A) = \frac{w_A}{p_x}$$

Hence, his expenditure function is:

$$e_A(p_x, p_y, u_A) = p_x u_A$$

Similarly, Beatrice's indirect utility function is

$$v_B(p_x, p_y, w_B) = x_B(p_x, p_y, w_B) = \frac{w_B}{p_y}$$

which yields an expenditure function of

$$e_B(p_x, p_y, u_B) = p_x u_B$$

- Hence, Alessandro's equivalent variation from the imposition of the tax is:

$$\begin{aligned} EV_A &= e(p^0, v_A(p^0, w_A^0)) - e_A(p^0, v_A(p^1, w_A^1)) \\ &= 150 - \frac{150}{3} = 100 \end{aligned}$$

while Beatrice's is

$$\begin{aligned} EV_B &= e(p^0, v_B(p^0, w_B^0)) - e_B(p^0, v_B(p^1, w_B^1)) \\ &= 150 - \frac{2 * 150}{2} = 0 \end{aligned}$$

Since Alessandro is the only individual who consumers spaghetti, he fully bears the tax burden (100), while Beatrice bears no burden at all since she does not consume spaghetti. Thus, since overall tax collection is 100, and originates only from Alessandro, the deadweight loss of the tax on Alessandro is zero,

$$DWL_A = 100 - 100 = 0$$

and similarly it is zero on Beatrice, since her $EV_B = 0$ and no tax collection originates from her

$$DWL_B = 0 - 0 = 0$$

Intuitively, since Alessandro is the only agent consuming spaghetti, a tax on spaghetti is equivalent to a lump-sum tax for him, and so this generates no deadweight loss. Since Beatrice does not consume spaghetti, the tax on spaghetti does not affect her consumption or her utility. But given the (arbitrary) equal division of wealth rule for this household, aggregate demand appears as if it is coming from a Cobb-Douglas utility function.

- Therefore, the sum of Alessandro's and Beatrice's deadweight losses is zero, while that of the representative consumer was positive, that is[1]

$$DWL_A + DWL_B = 0 \ < DWL$$

[1] For this example, a tax on spaghetti is distortionary and so generates a deadweight loss, but this only arises because wealth is equally divided between all members in the household. Thus, we cannot attach any normative significance to the "welfare" measures generated from this positive representative consumer. What appears to be a distortionary tax for the household is actually a non-distortionary tax at the individual level, equivalent to a lumpsum tax for Alessandro.