Homework # 2 EconS501 [Due on Sepetember 13th, 2021]

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1. **Consuming organic food.** Consider an individual with utility function

$$u(x_1, x_2) = \ln x_1 + x_2,$$

where x_1 and x_2 denote the amounts consumed of non-organic and organic goods, respectively. The prices of these goods are $p_1 > 0$ and $p_2 > 0$, respectively; and this individual's wealth is $w > 0$.

(a) Find this consumer's uncompensated demand for every good $x_i(p, w)$, where $i = \{1, 2\}$. [For compactness, we use p to denote the price vector $p \equiv (p_1, p_2)$.] Under which conditions the consumer demands positive amounts of both goods? Interpret your results.

- The tangency condition for this consumer, $MRS = \frac{p_1}{p_2}$, becomes

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{1}{x_1} = \frac{p_1}{p_2}$$

which simplifies to $p_1 x_1 = p_2$. Solving for x_1, we obtain the Walrasian demand for the non-organic good,

$$x_1(p, w) = \frac{p_2}{p_1}.$$

Substituting this Walrasian demand into the budget constraint $p_1 x_1 + p_2 x_2 = w$ yields

$$p_1 \underbrace{\frac{p_2}{p_1}}_{x_1} + p_2 x_2 = w.$$

Solving for x_2, we find the Walrasian demand for good 2 (organic good),

$$x_2(p, w) = \frac{w}{p_2} - 1$$

which is positive as long as $\frac{w}{p_2} > 1$, or if wealth w is sufficiently high, $w > p_2$. In this context, the consumer buys positive units of both organic and non-organic goods. Otherwise, the consumer only purchases a positive amount of the non-organic good $x_1(p, w) > 0$ but a zero amount of the organic good, $x_2(p, w) = 0$. Intuitively, this occurs when her income is relatively low.

- This result is due to the quasilinear utility function, leading the consumer to purchase strictly positive units of the good entering non-linearly (good 1) under all parameter values, but zero units of the good entering linearly (good 2) under relatively general parameter conditions.

(b) Find the indirect utility function, $v(p, w)$.

- Substituting the above Walrasian demands into the utility function gives the indirect utility function

$$\begin{aligned} v(p,w) &= \ln x_1(p,w) + x_2(p,w) \\ &= \ln\left(\frac{p_2}{p_1}\right) + \left(\frac{w}{p_2} - 1\right) \end{aligned}$$

(c) Find this consumer's expenditure function, $e(p, v)$, and her compensated demand for every good $h_i(p, w)$, where $i = \{1, 2\}$.

- *Expenditure function.* Solving for wealth w in the indirect utility function we found in part (a), $v(p, w)$, yields the expenditure function. Setting $v = v(p, w)$ and rearranging the indirect utility function, we obtain

$$v - \ln\left(\frac{p_2}{p_1}\right) + 1 = \frac{w}{p_2}$$

and solving for w, yields the expenditure function

$$e(p,v) = p_2\left[v - \ln\left(\frac{p_2}{p_1}\right) + 1\right]$$

- *Hicksian demands.* By Shepard's lemma, $h_1(p, v) = \frac{\partial e(p,v)}{\partial p_1}$, we can find Hicksian (compensated) demands by differentiating our above expenditure function with respect to the price of each good, as follows,

$$\begin{aligned} h_1(p,v) &= \frac{\partial e(p,v)}{\partial p_1} = \frac{p_2}{p_1}, \text{ and} \\ h_2(p,v) &= \frac{\partial e(p,v)}{\partial p_2} = v - \ln\left(\frac{p_2}{p_1}\right) \end{aligned}$$

Alternatively, we can also find Hicksian (compensated) demands by evaluating the Walrasian (uncompensated) demands at a wealth that coincides with the expenditure function, that is, $w = e(p, v)$, yielding

$$h_1(p,v) = x_1(p, e(p,v)) = \frac{p_2}{p_1}$$

for good 1 (since its Walrasian demand is independent of income, $x_1(p,w) = \frac{p_2}{p_1}$), and

$$h_2(p,v) = x_2(p,e(p,v)) = \frac{\overbrace{p_2\left[v - \ln\left(\frac{p_2}{p_1}\right) + 1\right]}^{w=e(p,v)}}{p_2} - 1$$

for good 2, which simplifies to

$$\begin{aligned} h_2(p,v) &= \left[v - \ln\left(\frac{p_2}{p_1}\right) + 1\right] - 1 \\ &= v - \ln\left(\frac{p_2}{p_1}\right) \end{aligned}$$

The Hicksian (compensated) demand for good 1 (organic) is independent of the utility level that the consumer targets in her expenditure minimization problem, v; but her Hicksian demand for good 2 (non-organic) is increasing in this utility level he seeks to target.

(d) Solve parts (a)-(c) of the exercise again, but considering that the consumer's utility function is now $u(x_1,x_2) = (x_1 - a_1)(x_2 - a_2)$, where parameters a_1 and a_2 are both weakly positive, $a_1, a_2 \geq 0$.

- *Finding Walrasian demand.* The tangency condition for this consumer, $MRS = \frac{p_1}{p_2}$, becomes

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{x_2 - a_2}{x_1 - a_1} = \frac{p_1}{p_2}$$

which simplifies to $p_1x_1 = p_1a_1 - p_2a_2 + p_2x_2$. Substituting this result into the budget constraint, $p_1x_1 + p_2x_2 = w$ yields

$$\underbrace{(p_1a_1 - p_2a_2 + p_2x_2)}_{p_1x_1} + p_2x_2 = w.$$

which simplifies to $p_1a_1 + p_2(a_2 - 2x_2) = w$. Solving for x_2, we obtain the Walrasian demand for good 2 (organic)

$$x_2(p,w) = \frac{w - p_1a_1 + p_2a_2}{2p_2}.$$

Inserting this result into the budget constraint, yields

$$p_1x_1 + p_2\underbrace{\left(\frac{w - p_1a_1 + p_2a_2}{2p_2}\right)}_{x_2(p,w)} = w$$

Solving for x_1, we find the Walrasian demand for good 1 (non-organic) to be

$$x_1(p, w) = \frac{w + p_1a_1 - p_2a_2}{2p_1}.$$

The Walrasian demand for good 2 (organic) is positive as long as $a_1 < \frac{w+p_2a_2}{p_1}$, whereas the Walrasian demand for good 1 (non-organic) is positive as long as $a_2 < \frac{w+p_1a_1}{p_2}$. Intuitively, the minimal amounts that the consumer needs to consume to obtain a positive utility level must be sufficiently small for her Walrasian demands to be positive.

- The Walrasian demand of every good i is increasing in the minimal amount that the consumer needs from that good a_i, but decreasing in the minimal amount that the consumer needs from the other good a_j. For instance, if the consumer does not need any positive amount of organic food but requires a large amount of non-organic food, $a_1 > 0$ but $a_2 = 0$, the above Walrasian demands collapse to

$$x_1(p, w) = \frac{w + p_1a_1}{2p_1} \quad \text{and} \quad x_2(p, w) = \frac{w - p_1a_1}{2p_2}$$

- *Indirect utility function.* Substituting the above Walrasian demands into the utility function gives the indirect utility function

$$\begin{aligned} v(p, w) &= (x_1(p, w) - a_1)(x_2(p, w) - a_2) \\ &= \left(\frac{w + p_1a_1 - p_2a_2}{2p_1} - a_1\right)\left(\frac{w - p_1a_1 + p_2a_2}{2p_2} - a_2\right) \\ &= \frac{(w - p_1a_1 - p_2a_2)^2}{4p_1p_2} \end{aligned}$$

- *Expenditure function.* Solving for wealth w in the indirect utility function we found in part (a), $v(p, w)$, yields the expenditure function. Setting $v = v(p, w)$, applying square roots on both sides, and rearranging the indirect utility function, we obtain

$$\sqrt{v} = \frac{w - p_1a_1 - p_2a_2}{2\sqrt{p_1p_2}}$$

and solving for w, yields the expenditure function

$$e(p, v) = 2\sqrt{vp_1p_2} + p_1a_1 + p_2a_2$$

- *Hicksian demands.* By Shepard's lemma, $h_1(p, v) = \frac{\partial e(p,v)}{\partial p_1}$, we can find Hicksian (compensated) demands by differentiating our above expenditure function with respect to the price of each good, as follows,

$$\begin{aligned} h_1(p, v) &= \frac{\partial e(p, v)}{\partial p_1} = a_1 + \sqrt{v\frac{p_2}{p_1}}, \text{ and} \\ h_2(p, v) &= \frac{\partial e(p, v)}{\partial p_2} = va_2 + \sqrt{v\frac{p_1}{p_2}}. \end{aligned}$$

2. **Composite goods.** Consider a consumer with utility function $u(x_1, x_2, x_3) = x_1x_2x_3$, and income w.

   (a) Set up the consumer's utility maximization problem and find the Walrasian demands for each good.

   - The consumer solves

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & x_1x_2x_3 \\ \text{s.t.} \quad & p_1x_1 + p_2x_2 + p_3x_3 \leq w \end{aligned}$$

   Setting up the Lagrangian, we write

$$L = x_1x_2x_3 + \lambda(w - p_1x_1 - p_2x_2 - p_3x_3)$$

   which yields the first-order conditions

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= x_2x_3 - \lambda p_1 = 0 \\ \frac{\partial L}{\partial x_2} &= x_1x_3 - \lambda p_2 = 0 \\ \frac{\partial L}{\partial x_3} &= x_1x_2 - \lambda p_3 = 0 \\ \frac{\partial L}{\partial \lambda} &= w - p_1x_1 - p_2x_2 - p_3x_3 = 0 \end{aligned}$$

In the case of interior solutions, solving for λ yields the following relations

$$\begin{aligned}\frac{x_2}{x_1} &= \frac{p_1}{p_2} \Longleftrightarrow \frac{p_2 x_2}{p_1} = x_1 \\ \frac{x_3}{x_2} &= \frac{p_2}{p_3} \Longleftrightarrow x_3 = \frac{p_2 x_2}{p_3} \\ \frac{x_2 x_3}{x_1 x_2} &= \frac{p_1}{p_3}\end{aligned}$$

Substituting the above conditions into the budget constraint gives

$$\begin{aligned}p_1 x_1 + p_2 x_2 + p_3 x_3 &= \\ p_1 \underbrace{\frac{p_2 x_2}{p_1}}_{x_1} + p_2 x_2 + p_3 \underbrace{\frac{p_2 x_2}{p_3}}_{x_3} &= w\end{aligned}$$

Finally, solving for x_2 yields the Walrasian demand for good x_2,

$$x_2(w, p_1, p_2, p_3) = \frac{w}{3p_2}.$$

Similar manipulations gives the Walrasian demands for goods x_1 and x_3,

$$\begin{aligned}x_1(w, p_1, p_2, p_3) &= \frac{w}{3p_1} \\ x_3(w, p_1, p_2, p_3) &= \frac{w}{3p_3}\end{aligned}$$

(b) Let $x_1 + \frac{p_2}{p_1} x_2 = x_c$ denote the units of a composite good. Set up the consumer's utility maximization problem again, but now in terms of the composite good x_c. Find the Walrasian demand function for the composite good x_c.

- Since $x_1 + \frac{p_2}{p_1} x_2 = x_c$, we can express x_1 as $x_1 = x_c - \frac{p_2}{p_1} x_2$. The consumer then solves

$$\max_{x_1, x_2, x_3} \overbrace{\left(x_c - \frac{p_2}{p_1} x_2\right)}^{x_1} x_2 x_3$$
$$\text{s.t.} \quad p_1 x_c + p_3 x_3 \leq w$$

Setting up the Lagrangian, we write

$$L = (x_c - \frac{p_2}{p_1} x_2) x_2 x_3 + \lambda(w - p_1 x_c - p_3 x_3)$$

which yields the first-order conditions

$$\begin{aligned}
\frac{\partial L}{\partial x_c} &= x_2 x_3 - \lambda p_1 = 0 \\
\frac{\partial L}{\partial x_2} &= -\left(\frac{p_2}{p_1}\right) x_2 x_3 + \left(x_c - \frac{p_2}{p_1} x_2\right) x_3 = 0 \\
\frac{\partial L}{\partial x_3} &= \left(x_c - \frac{p_2}{p_1} x_2\right) x_2 - \lambda p_3 = 0 \\
\frac{\partial L}{\partial \lambda} &= w - p_1 x_c - p_3 x_3 = 0
\end{aligned}$$

From the second first-order condition we obtain

$$x_2 = x_c \frac{p_1}{2p_2}$$

Combining first and third first-order conditions gives

$$x_3 = \frac{\left(x_c - \frac{p_2}{p_1} x_2\right) p_1}{p_3} = x_c \frac{p_1}{2p_3}$$

Substituting the expression for x_3 into the budget constraint yields the Walrasian demand for good x_c

$$x_c = \frac{2w}{3p_1}$$

which entails that the Walrasian demands for goods 2 and 3 are

$$\begin{aligned}
x_2 &= x_c \frac{p_1}{2p_2} = \frac{2w}{2p_1} \frac{p_1}{2p_2} = \frac{w}{3p_2} \\
x_3 &= x_c \frac{p_1}{2p_3} = \frac{2w}{3p_1} \frac{p_1}{2p_3} = \frac{w}{3p_3}
\end{aligned}$$

(c) Show that the Walrasian demands you found in parts (a) and (b) are equivalent.

- As shown in part (b), the Walrasian demands for good 2 and 3 coincide with those found in part (a). Regarding the Walrasian demand for good 1, we can also confirm this coincidence, as follows

$$\begin{aligned}
x_1 &= x_c - \frac{p_2}{p_1} x_2 \\
&= \frac{2w}{3p_1} - \frac{p_2}{p_1} \underbrace{\frac{w}{3p_2}}_{x_2} \\
&= \frac{w}{3p_1}.
\end{aligned}$$

3. Consider a consumer with quasilinear utility function $u(x, y, q) = v(x, q) + y$, where x denotes units of good x, q represents its quality, and y reflects the numeraire good (whose price is normalized to 1). The price of good x is $p > 0$, and the consumer's wealth is $w > 0$. Assume that $v_x, v_q > 0$ and $v_{xx} \leq 0$.

(a) Set up the consumer's utility maximization problem.

- Solving for y in the budget constraint $px + y = w$, i.e., $y = w - px$, the problem can be written as the following unconstrained problem with x as the only choice variable.

$$\max_{x \geq 0} v(x, q) + \overbrace{(w - px)}^{y}$$

Differentiating with respect to x, we obtain

$$v_x(x(p, q), q) = p$$

where $x(p, q)$ denotes the Walrasian demand for good x. In words, the above equation indicates that the consumer increases his purchases of good x until the point where his marginal utility for additional units coincides with the good's price.

(b) Show that the Walrasian demand $x(p, q)$ is: (1) decreasing in p; and (2) increasing in q if $v_{xq} > 0$. Interpret your results.

- *Price.* Differentiating the equation we found in part (a), $v_x(x(p, q), q) = p$, with respect to p, yields

$$v_{xx}\frac{\partial x(p, q)}{\partial p} = 1$$

where we used the Chain rule. Solving for $\frac{\partial x(p,q)}{\partial p}$, we find that

$$\frac{\partial x(p, q)}{\partial p} = \frac{1}{v_{xx}}.$$

Since $v_{xx} \leq 0$ by definition, $\frac{\partial x(p,q)}{\partial p}$ is negative; as required. Intuitively, the law of demand holds, i.e., a more expensive good x decreases the consumer's purchases of this good. (Recall that we only assumed that function v is increasing and concave in good x, and that it is increasing in the good's quality q.)

- *Quality.* Similarly, differentiating $v_x(x(p, q), q) = p$, with respect to q, we find that

$$v_{xx}\frac{\partial x(p, q)}{\partial q} + v_{xq} = 0.$$

Solving for $\frac{\partial x(p,q)}{\partial q}$, we find that

$$\frac{\partial x(p,q)}{\partial q} = -\frac{v_{xq}}{v_{xx}}.$$

Since $v_{xx} \leq 0$ by definition, $\frac{\partial x(p,q)}{\partial q}$ is positive if $v_{xq} > 0$; as required. Otherwise, $\frac{\partial x(p,q)}{\partial q}$ becomes negative.

Intuitively, the consumer demands more units of good x when its quality increases if quality increases the marginal utility of good x, i.e., $v_{xq} > 0$. If, instead, a higher quality were to decrease the marginal utility that the consumer obtains from good x, $v_{xq} < 0$, then a higher quality would induce him to reduce his purchases, i.e., $\frac{\partial x(p,q)}{\partial q} < 0$. Finally, note that if quality has no effect on the marginal utility he enjoys from the good, $v_{xq} = 0$, his purchases would be also unaffected by q, i.e., $\frac{\partial x(p,q)}{\partial q} = 0$.

(c) Assume in this part of the exercise that $v_{xq} > 0$ so that $\frac{\partial x(p,q)}{\partial q} > 0$. We say that a Walrasian demand $x(p,q)$ is supermodular in (p,q) if the following property holds

$$\underbrace{x(p,q)\frac{\partial^2 x(p,q)}{\partial p \partial q}}_{\text{First term}} - \underbrace{\underbrace{\frac{\partial x(p,q)}{\partial p}}_{(-)\text{ from part (b)}} \underbrace{\frac{\partial x(p,q)}{\partial p}}_{(+)\text{ from part (b)}}}_{\text{Second term, }+} > 0.$$

From part (b) we know that $\frac{\partial x(p,q)}{\partial p} < 0$ and that $\frac{\partial x(p,q)}{\partial q}$ is positive. Therefore, for Walrasian demand $x(p,q)$ to be supermodularity we only need that the cross-partial $\frac{\partial^2 x(p,q)}{\partial p \partial q}$ is either positive, entailing an unambigous expression above, or not very negative, so the positive second term offsets the potentially negative first term. Show that supermodularity holds if $v_{xx}v_{xq} + x(v_{xxx}v_{xq} - v_{xxq}v_{xx}) < 0$. Interpret your results.

- Differentiating our results from part (a) twice with respect to p, we find

$$\left(v_{xxx}\frac{\partial x(p,q)}{\partial q} + v_{xxq}\right)\frac{\partial x(p,q)}{\partial p} + v_{xx}\frac{\partial^2 x(p,q)}{\partial p \partial q} = 0.$$

Therefore, the condition for supermodularity in the Walrasian demand entails

$$\underbrace{\frac{1}{v_{xx}^3}}_{-}\left[x(p,w)v_{xxx}v_{xq} - x(p,w)v_{xxq}\underbrace{v_{xx}}_{-} + v_{xq}\underbrace{v_{xx}}_{-}\right] > 0.$$

Since $v_{xx} \leq 0$ by assumption, we find that the above expression is positive as long as

$$v_{xx}v_{xq} + x(p,w)(v_{xxx}v_{xq} - v_{xxq}v_{xx}) < 0.$$

- Intuitively, this condition holds if the marginal utility of good x, v_x satisfies the gross complementarity condition in consumer theory. We discussed the gross complementarity condition in the context of the utility of good x, i.e., $v_x v_q + x(p,w)(v_{xx}v_q - v_{xq}v_x) < 0$ in this setting, while the above expression applies it to the marginal utility of x, v_x.

4. **Decomposing income elasticity.** Consider utility function $u(x,y)$, where x and y represent the units of two goods. Assume that $u(\cdot)$ is twice continuously differentiable, strictly increasing and concave in both of its arguments, x and y. Assuming that the consumer's wealth is given by $w > 0$, and that he faces a price vector $p = (p_x, p_y) >> 0$, denote his indirect utility function as $v(p,w)$.

   (a) Use the indirect utility function $v(p,w)$ to find the consumer willingness to pay for good y.

   - The indirect utility function can be found by solving the consumer's utility maximization problem subject to her budget constraint as follows:

   $$v(p,w,y) = \max u(x,y) \quad \text{s.t.} \quad p_x x + p_y y \leq w.$$

   Define the marginal rate of substitution between income and good y, $MRS_{y,w}$, such that:

   $$MRS_{y,w} = \frac{v_y}{v_w}$$

   where $v_y = \frac{\partial v}{\partial y}$ and $v_w = \frac{\partial v}{\partial w}$. Then, define the willingness to pay for good y as the product $WTP = MRS_{y,w} \times y$.

   (b) Identify under which condition is this willingness to pay for good y increasing or decreasing in income, w. Interpret.

   - To examine how WTP for good y varies with income, w, we need to determine the income effect $\frac{\partial WTP}{\partial w}$. It may be helpful to estimate the value of the income elasticity of WTP, which is defined as:

   $$\varepsilon_{WTP}^{w} = \frac{\frac{\partial WTP}{WTP}}{\frac{\partial w}{w}} = \frac{\partial WTP}{\partial w}\frac{w}{WTP}$$

   Since $w > 0$, $y > 0$ and $WTP > 0$, we obtain that and $\frac{w}{WTP} > 1$. Therefore, ε_{WTP}^{w} has the same sign as $\frac{\partial WTP}{\partial w}$. Since $WTP = MRS_{y,w} \times y$ by definition,

$\frac{\partial WTP}{\partial w}$ has the same sign as $\frac{\partial MRS_{y,w}}{\partial w}$. Let us next find this derivative

$$\frac{\partial MRS_{y,w}}{\partial w} = \frac{v_{wy}v_w - v_{ww}v_y}{v_w^2}$$

where $v_w > 0$, $v_y > 0$, and by assumption $v_{ww} < 0$. Hence, the sign of $\frac{\partial MRS_{y,w}}{\partial w}$ depends on the sign of the cross derivative v_{wy}, which intuitively indicates the interaction between income and good y in the utility function. Hence, we can identify two cases:

- $\frac{\partial WTP}{\partial w} < 0$, implying that the willingness to pay for good y decreases with income, only if income and good y are regarded as substitutes or independent by the consumer, i.e., $v_{wy} < 0$ or $v_{wy} = 0$.
- The opposite case, $\frac{\partial WTP}{\partial w} > 0$, indicating that the willingness to pay for good y increases with income, can occur: (1) under complementarity (i.e., $v_{wy} > 0$); and (2) under substitutability ($v_{wy} < 0$ if, in addition, the numerator of $\frac{\partial MRS_{y,w}}{\partial w}$ is negative, that is $|v_{wy}v_w| < |v_{ww}v_y|$).