Homework 1 - EconS 501 (Wednesday, September 1st 2021)

1. Consider that your preference relation over three bundles, x_1, x_2, and x_3, satisfies

$$x_1 \succ x_2$$
$$x_2 \succ x_3$$
$$x_3 \succ x_1$$

   (a) Show that you can be wiped out of your wealth w, where $w > 0$. (Hint: Begin with x_3.)

   - First, let me begin with $x_2 \succ x_3$. I am willing to pay an amount α, where $\alpha > 0$, to exchange x_3 for x_2, since $x_2 \succ x_3$, ending up with wealth $w - \alpha$. The monetary amount $\alpha > 0$ can be as small as necessary to induce me to exchange x_3 for x_2.
   - Second, since $x_1 \succ x_2$, I am willing to pay an amount β, where $\beta > 0$, to exchange x_2 for x_1, ending up with wealth $w - \alpha - \beta$. (The monetary amount β can coincide with α, $\beta = \alpha$, or differ, $\beta \neq \alpha$, without affecting our final result.)
   - Third, since $x_3 \succ x_1$, I am willing to pay an amount γ, where $\gamma > 0$, to exchange x_1 for x_3, ending up with wealth $w - \alpha - \beta - \gamma$. (The monetary amount γ can coincide with α, β, or none of them, without affecting our final result.)
   - Defining the sum of monetary amounts I paid so far, $y \equiv \alpha + \beta + \gamma$, I end up with my original bundle, x_3, but my wealth is reduced from w to $w - y$. Repeating the above three steps for n rounds, my wealth is wiped out, that is, we can find a number of rounds $\overline{n} \in \mathbb{N}$ such that

$$w - (\overline{n} + 1)\, y < 0 < w - \overline{n} y$$

     meaning that, at round $\overline{n}$, my wealth is still positive; but going through another round, $\overline{n} + 1$, will leave me with a negative wealth.

   (b) Consider an individual with a preference relation that violates rationality because his preferences are incomplete or intransitive. Discuss.

   - A rational preference relation must be complete and transitive. Suppose that my preference relation is, instead:

– incomplete, then there exists a pair of alternatives x and y in set X which I cannot compare, implying that my preference relation is not well defined.

– intransitive, then there exists a "money pump" as in part (a) that can eliminate all my wealth after finite rounds of exchanges.

Thus, preference relation must be complete and transitive in order to be rational.

2. Consider the following preference relation defined in $X = \mathbb{R}^2_+$. A bundle (x_1, x_2) is weakly preferred to another bundle (y_1, y_2), i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if

$$\max\{x_1 + 2x_2, 2x_1 + x_2\} \geq \min\{y_1 + 2y_2, 2y_1 + y_2\}$$

(a) For any given bundle (y_1, y_2), draw the upper contour set, the lower contour set, and the indifference set of this preference relation (take point $(3, 1)$). Interpret.

- Take a bundle $(3, 1)$. Then,

$$\min\{3 + 2 \times 1, 3 \times 2 + 1 \times 1\} = \min\{5, 7\} = 5.$$

The upper contour set of this bundle is given by

$$\begin{aligned} UCS(3,1) &= \{(x_1, x_2) \succsim (3, 1)\} \\ &= \{\max\{x_1 + 2x_2, 2x_1 + x_2\} \geq 5\} \end{aligned}$$

which is graphically represented by all those bundles in $\mathbb{R}^2_+$ which are strictly above *both* lines $x_1 + 2x_2 = 5$ and $2x_1 + x_2 = 5$. That is, for all (x_1, x_2) strictly above both lines

$$x_2 = \frac{5}{2} - \frac{1}{2}x_1 \text{ and } x_2 = 5 - 2x_1.$$

- On the other hand, the lower contour set is defined as

$$\begin{aligned} LCS(3,1) &= \{(3, 1) \succsim (x_1, x_2)\} \\ &= \{7 \geq \min\{x_1 + 2x_2, 2x_1 + x_2\}\}, \end{aligned}$$

which is graphically represented by all bundles (x_1, x_2) weakly below the maximum of the lines described above. For instance, bundle $(y_1, y_2) = (3.5, 0)$, which lies on the horizontal axis, implies

$$\max\{3.5 + 2 \cdot 0, 2 \cdot 3.5 + 1 \cdot 0\} = \max\{3.5, 7\} = 7$$

thus implying that this consumer weakly prefers bundle $(x_1, x_2) = (3, 1)$ than $(y_1, y_2) = (3.5, 0)$. A similar argument applies to all other bundles lying above $x_2 = \frac{7}{2} - \frac{1}{2}x_1$ and above $x_2 = 7 - 2x_1$, where bundle $(3.5, 0)$ also belongs. Similarly, bundles such as $(0, 3.5)$ yield

$$\max\{1 \cdot 0 + 2 \cdot 3.5, 2 \cdot 0 + 1 \cdot 3.5\} = \max\{7, 3.5\} = 7,$$

which implies that the consumer also weakly prefers bundle $(3, 1)$ to $(0, 3.5)$. An analogous argument applies to all bundles above line $x_2 = \frac{7}{2} - \frac{1}{2}x_1$ and $x_2 = 7 - 2x_1$.

Finally, those bundles for which the UCS and LCS overlap are those in IND of bundle (3,1).

(b) Check if this preference relation satisfies: (i) completeness and (ii) transitivity.

- *Completeness.* First, note that both of the elements in the $\max\{\cdot\}$ operator are real numbers, i.e., $(x_1 + 2x_2) \in \mathbb{R}_+$ and $(2x_1 + x_2) \in \mathbb{R}_+$, thus implying that the maximum

$$\max\{x_1 + 2x_2, 2x_1 + x_2\} = a$$

exists and it is also a real number, $a \in \mathbb{R}_+$. Similarly, the minimum

$$\min\{y_1 + 2y_2, 2y_1 + y_2\} = b$$

exists and is a real number, $b \in \mathbb{R}_+$. Therefore, we can easily compare a and b, obtaining that either $a \geq b$, which implies $(x_1, x_2) \succsim (y_1, y_2)$; or $a \leq b$, which implies $(y_1, y_2) \succsim (x_1, x_2)$, or both, $a = b$, which entails $(x_1, x_2) \sim (y_1, y_2)$. Hence, the preference relation is complete.

- *Transitivity.* We need to show that, for any three bundles (x_1, x_2), (y_1, y_2) and (z_1, z_2) such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ implies

$$a \equiv \max\{x_1 + 2x_2, 2x_1 + x_2\} \geq \min\{y_1 + 2y_2, 2y_1 + y_2\} \equiv b$$

and $(y_1, y_2) \succsim (z_1, z_2)$ implies that

$$b \equiv \max\{y_1 + 2y_2, 2y_1 + y_2\} \geq \min\{z_1 + 2z_2, 2z_1 + z_2\} \equiv c$$

In order to guarantee that b is the $\max\{y_1 + 2y_2, 2y_1 + y_2\}$ *and* the $\min\{y_1 + 2y_2, 2y_1 + y_2\}$ we require that $y_1 = y_2$. Combining both conditions we have that $a \geq b \geq c$, which implies that $a \geq c$. Hence, we have that

$$\max\{x_1 + 2x_2, 2x_1 + x_2\} \geq \min\{z_1 + 2z_2, 2z_1 + z_2\}$$

and thus $(x_1, x_2) \succsim (z_1, z_2)$, implying that this preference relation is transitive.

3. Explain transitivity in preference relations. Provide an example (different to the examples discussed in class) where this property is not satisfied and discuss the consequences of intransitive preferences.

4. Explain monotonicity and strong monotonicity in preference relations, and compare them. Provide an example where a bundle x is (strictly) preferred to bundle y when preferences satisfy strong monotonicity, but x is not necessarily preferred to y under monotonicity.

   - *Monotonicity* states that increasing the amount of some commodities cannot hurt, and increasing the amount of all commodities is strictly preferred. Formally, if we take bundle $y \in \mathbb{R}^L$ and weakly increase all k components, so that we generate a new bundle $x \in \mathbb{R}^L$ satisfying $x_k \geq y_k$ for all k, then an individual with monotonic preferences would prefer the newly created bundle to the original bundle, i.e., $x \succsim y$. (Note that this implies that at least one component of the bundle has been strictly increased while the remaining components can be left unaffected.) In addition, if we strictly increase the amount of all components in bundle y, this individual would strictly prefer the new bundle, i.e., if $x_k > y_k$ for all k, then $x \succ y$.
   - *Strong monotonicity.* On the other hand, strong monotonicity states that the consumer is strictly better off with additional amounts of any commodity. That is, if we strictly increase the amount of at least one commodity, the consumer strictly prefers the newly created bundle x to his original bundle y. That is, if $x_k \geq y_k$ for all good k and $x \neq y$, then $x \succ y$. (Note that this implies that $x_j > y_j$ for at least one commodity j, since otherwise both bundles would coincide.)
   - *Comparison.* Then, a consumer's preference relation can satisfy monotonicity (if additional amounts of one of his commodity do not harm his utility), but does not need to satisfy strong monotonicity (since for that to occur, he would need to become strictly better off as a consequence of the additional amounts in one of his commodities). However, if a consumer's preferences satisfy strong monotonicity,

they must also satisfy monotonicity. That is why strong monotonicity is a more restrictive ("stronger") assumption on preferences than monotonicity.

- *Example*: Consider bundles $x = (1, 2)$ and $y = (1, 1)$. If preferences satisfy strong monotonicity, $x \succ y$ since the second component in bundle x is higher than the corresponding component in y, i.e., $x_j \geq y_j$ for some good j. However, if preferences only satisfy monotonicity, we cannot state that $x \succ y$ (strictly), since $x_k > y_k$ does not hold for all k commodities.