Solution - Recitation 1, Fall 2021

1. **Lexicographic preference relation.** Let us define a lexicographic preference relation in a consumption set $X \times Y$, as follows:

$$(x_1, x_2) \succsim (y_1, y_2) \text{ if and only if } \begin{cases} x_1 > y_1, \text{ or if} \\ x_1 = y_1 \text{ and } x_2 \geqslant y_2 \end{cases} \quad (1)$$

Intuitively, the consumer prefers bundle x to y if the former contains more units of the first good than the latter, i.e., $x_1 > y_1$. However, if both bundles contain the same amounts of good 1, $x_1 = y_1$, the consumer ranks bundle x above y if the former has more units of good 2 than the latter, i.e., $x_2 \geq y_2$. For simplicity, assume that both components have been normalized to $X = [0, 1]$ and $Y = [0, 1]$.

(a) Show that the lexicographic preference relation satisfies rationality (i.e., it is complete and transitive).

1. *Completeness.* By definition, $\succsim$ is a complete preference relation if for all bundles (x_1, x_2),$(y_1, y_2) \in \mathbb{R}^2$, either $(x_1, x_2) \succsim (y_1, y_2)$, or $(y_1, y_2) \succsim (x_1, x_2)$, or both. Hence, we need to show that

$$(x_1, x_2) \not\succsim (y_1, y_2) \implies (y_1, y_2) \succsim (x_1, x_2)$$

Indeed, note that $(x_1, x_2) \not\succsim (y_1, y_2)$ can be expressed as

$$(x_1, x_2) \not\succsim (y_1, y_2) \text{ if } \begin{cases} y_1 \geq x_1, \text{ and if} \\ y_1 \neq x_1 \text{ or } y_2 > x_2 \end{cases} \quad (2)$$

Expression (2) describes that bundle (y_1, y_2) contains weakly more units of good 1 than (x_1, x_2) does, thus implying that a consumer with a lexicographic preference relation weakly prefers (y_1, y_2) to (x_1, x_2), i.e., $(y_1, y_2) \succsim (x_1, x_2)$. Therefore, we have shown that $(x_1, x_2) \not\succsim (y_1, y_2)$ implies $(y_1, y_2) \succsim (x_1, x_2)$. Hence, the preference relation is complete.

2. *Transitivity.* Let us take three bundles (x_1, x_2),(y_1, y_2) and $(z_1, z_2) \in \mathbb{R}^2$ with $(x_1, x_2) \succsim (y_1, y_2)$:

$$(x_1, x_2) \succsim (y_1, y_2) \text{ if and only if } \begin{cases} x_1 > y_1, \text{ or if} \\ x_1 = y_1 \text{ and } x_2 \geqslant y_2 \end{cases}$$

and $(y_1, y_2) \succsim (z_1, z_2)$, that is

$$(y_1, y_2) \succsim (z_1, z_2) \text{ if and only if } \begin{cases} y_1 > z_1, \text{ or if} \\ y_1 = z_1 \text{ and } y_2 \geqslant z_2 \end{cases}$$

Hence, we need to check for transitivity in the four possible cases in which $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2)$.

a) If $x_1 > y_1$, and $y_1 > z_1$, then by the transitivity of the "greater than or equal" operator ($\geq$), we obtain $x_1 > z_1$. As we know that $x_1 > z_1$ implies $(x_1, x_2) \succsim (z_1, z_2)$, then transitivity holds in this case.

b) If ($x_1 = y_1$ and $x_2 \geqslant y_2$) and ($y_1 = z_1$ and $y_2 \geqslant z_2$), then ($x_1 = z_1$ and $x_2 \geqslant z_2$). In addition, we know that ($x_1 = z_1$ and $x_2 \geqslant z_2$) implies $(x_1, x_2) \succsim (z_1, z_2)$, which validates transitivity.

c) If $x_1 > y_1$, and ($y_1 = z_1$ and $y_2 \geqslant z_2$), then $x_1 > z_1$. As we know that $x_1 > z_1$ implies $(x_1, x_2) \succsim (z_1, z_2)$. Transitivity holds in this case as well.

d) If $y_1 > z_1$ and ($x_1 = y_1$ and $x_2 \geqslant y_2$), then $x_1 > z_1$, and we know that $x_1 > z_1$ implies $(x_1, x_2) \succsim (z_1, z_2)$, entailing that transitivity holds in this case as well. We have then checked all four cases under which $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2)$ may arise, and in all of them we obtained $(x_1, x_2) \succsim (z_1, z_2)$, confirming that this preference relation is transitive. Therefore, since the preference relation is complete and transitive, we can conclude that it is rational.

(b) Show that the lexicographic preference relation $\succsim$ *cannot* be represented by a utility function $u : X \times Y \to \mathbb{R}$.

- Let us work by contradiction. So, let us suppose that there is a utility function $u(\cdot)$ representing this lexicographic preference relation $\succsim$. Then, for any $x_1 \in X$, the pair $(x_1, 1)$ is strictly preferred to the pair $(x_1, 0)$, i.e., $(x_1, 1) \succ (x_1, 0)$. If there is a utility function $u(\cdot)$ representing this preference relation, then we must have that

$$(x_1, 1) \succ (x_1, 0) \Longleftrightarrow u(x_1, 1) > u(x_1, 0)$$

On the other hand, from the Archimedean property, we know that we can pick a rational number $r(x_1)$ such that it lies in between $u(x_1, 1)$ and $u(x_1, 0)$.

$$u(x_1, 1) > r(x_1) > u(x_1, 0)$$

Let us take any $x_1, x_2 \in X$, and let us suppose without loss of generality that

$x_1 > x_2$. Similarly to our above result, we then have that

$$u(x_2, 1) > r(x_2) > u(x_2, 0)$$

And since $x_1 > x_2$, we have that

$$u(x_1, 1) > r(x_1) > u(x_1, 0) > u(x_2, 1) > r(x_2) > u(x_2, 0)$$

which implies

$$r(x_1) > r(x_2)$$

Then, $r(\cdot)$ provides a one-to-one function from the set of real numbers, $\mathbb{R}$ (which is uncountable) to the set of rational numbers, $\mathbb{Q}$, which is countable. But this is a mathematical impossibility.[1] Thus, we conclude that there can be no utility function representing the lexicographic preferences when they are defined over a continuous set $X \times Y$, where $X = [0, 1]$ and $Y = [0, 1]$.

(c) Assume now that this preference relation is defined on a *finite* consumption set $X = X_1 \times X_2$, where $X_1 = \{x_{11}, x_{12}, ..., x_{1n}\}$ and $X_2 = \{x_{21}, x_{22}, ..., x_{2m}\}$. [*Hint*: You can define a function $N_i(x_{ij})$ as the number of elements in sequence X_i prior to element x_{ij}; that is,

$$N_i\left(x_{ij}\right) = \#\left\{y \in X_i | y < x_{ij}\right\}.$$

Then define a utility function

$$u(y_1, y_2) = mN_1(y_1) + N_2(y_2), \text{ where } m > 0,$$

and for any pair $(y_1, y_2) \in X_1 \times X_2$.]

1. Let us first define a function $N_i(x_{ij})$ as the number of elements in sequence X_i prior to element x_{ij}:

$$N_i\left(x_i\right) = \#\left\{y \in X_i | y < x_{ij}\right\}, \text{ where } X_i = \left\{x_{i1}, x_{i2}, ..., x_{in}\right\}.$$

Then, we define a utility function $u(y_1, y_2) = mN_1(y_1) + N_2(y_2)$ for any pair $(y_1, y_2) \in X_1 \times X_2$. In order to show that this utility function indeed represents the lexicographic preference relation (when consumption sets are finite), we

[1] For a review of real and rational numbers, see, for instance, Simon and Blume's *Mathematics for Economists*, pp. 848-849

need to show the usual two lines of implication:

$$(y_1, y_2) \succsim (z_1, z_2) \Longrightarrow u(y_1, y_2) \geq u(z_1, z_2), \text{ and}$$

$$(y_1, y_2) \succsim (z_1, z_2) \Longleftarrow u(y_1, y_2) \geq u(z_1, z_2)$$

2. Let us first show that $(y_1, y_2) \succsim (z_1, z_2) \Longrightarrow u(y_1, y_2) \geq u(z_1, z_2)$. In order to show this result, we need that

$$\begin{cases} y_1 > z_1, \text{ or} \\ y_1 = z_1 \text{ and } y_2 \geqslant z_2 \end{cases} \quad \text{implies } mN_1(y_1) + N_2(y_2) \geq mN_1(z_1) + N_2(z_2)$$

Hence, we first need to check if this inequality is satisfied when $y_1 > z_1$, and when ($y_1 = z_1$ and $y_2 \geqslant z_2$).

a) Let us first check if $y_1 > z_1$ implies $mN_1(y_1)+N_2(y_2) \geq mN_1(z_1)+N_2(z_2)$. Alternatively, we can rewrite this inequality as

$$m\underbrace{[N_1(y_1) - N_1(z_1)]}_{a} + \underbrace{[N_2(y_2) - N_2(z_2)]}_{b} \geq 0 \tag{1}$$

Let us analyze if this expression can ever be negative (we will examine the infimum values) by separately evaluating the infimum of terms (a) and (b). Regarding term (a), we know that, if $y_1 > z_1$,

$$\inf [N_1(y_1) - N_1(z_1)] = k - (k - 1) = 1,$$

$$\text{since } N_1(y_1) > N_1(z_1) \text{ given that } y_1 > z_1$$

and hence, $\inf [m [N_1(y_1) - N_1(z_1)]] = m$. Thus, $m [N_1(y_1) - N_1(z_1)] \geq m$, and term (a) in expression (1) is always weakly above m. Let us now focus on term (b) of expression (1):[2]

$$\inf [N_2(y_2) - N_2(z_2)] = \inf N_2(y_2) - \sup N_2(z_2) = 0 - (m - 1) = 1 - m$$

Intuitively, the result $\inf N_2(y_2) = 0$ implies that there are no elements prior to y_2 (that is, y_2 is the first term of the sequence); in contrast, $\sup N_2(z_2) = m - 1$ means that z_2 is the last element in the sequence of length m, and hence all other $m - 1$ elements in the sequence were located prior to z_2. Hence, $N_1(y_1) - N_1(z_1) \geq 1 - m$, and thus term (b)

[2]Note that we are not imposing any conditions on y_2 and z_2, since we only assumed that $y_1 > z_1$.

in expression (1) always lies above $1-m$. Combining the results of the first and second term of the infimum of expression (1), we can conclude that

$$m\,[N_1(y_1)-N_1(z_1)]+[N_2(y_2)-N_2(z_2)]\geq m-(1-m)=1$$

which is clearly above 0. Recall that we needed to show that

$$m\,[N_1(y_1)-N_1(z_1)]+[N_2(y_2)-N_2(z_2)]\geq 0.$$

Therefore, $y_1>z_1$ indeed implies $u(y_1,y_2)\geq u(z_1,z_2)$.

b) Let us now check that ($y_1=z_1$ and $y_2\geqslant z_2$) also implies $mN_1(y_1)+N_2(y_2)\geq mN_1(z_1)+N_2(z_2)$. Alternatively, we can rewrite this inequality as

$$m\,[N_1(y_1)-N_1(z_1)]+[N_2(y_2)-N_2(z_2)]\geq 0$$

First, note that $y_1=z_1$ implies that $N_1(y_1)=N_1(z_1)$. Second, note that $y_2\geqslant z_2$ implies that $N_2(y_2)\geq N_2(z_2)$. Therefore, the above inequality becomes

$$0+\underbrace{[N_2(y_2)-N_2(z_2)]}_{\geq 0}\geq 0$$

which confirms what we needed to show. Hence, ($y_1=z_1$ and $y_2\geqslant z_2$) indeed implies $u(y_1,y_2)\geq u(z_1,z_2)$.

3. Let us now show the opposite direction of implication, i.e., $(y_1,y_2)\succsim(z_1,z_2)\Longleftarrow u(y_1,y_2)\geq u(z_1,z_2)$. First, note that if $u(y_1,y_2)\geq u(z_1,z_2)$, then it must be that $mN_1(y_1)+N_2(y_2)\geq mN_1(z_1)+N_2(z_2)$. Rearranging, we obtain

$$m\,[N_1(y_1)-N_1(z_1)]+[N_2(y_2)-N_2(z_2)]\geq 0$$

Then, note that this inequality can be positive for two different reasons: (1) because $N_1(y_1)>N_1(z_1)$, which implies $y_1>z_1$; or because (2) $N_1(y_1)=N_1(z_1)$ and $N_2(y_2)\geq N_2(z_2)$, which implies $y_1=z_1$ and $y_2\geqslant z_2$. And we know that, by definition, these two cases describe the lexicographic preference relation

$$(y_1,y_2)\succsim(z_1,z_2)\ \text{ if and only if }\ \begin{cases} y_1>z_1, \ \text{or if} \\ y_1=z_1 \text{ and } y_2\geqslant z_2 \end{cases}$$

Hence, $(y_1,y_2)\succsim(z_1,z_2)\Longleftarrow u(y_1,y_2)\geq u(z_1,z_2)$. Since we have shown this implication in both directions, then we have confirmed that this utility function indeed represents the lexicographic preference relation

$$(y_1, y_2) \succsim (z_1, z_2) \Longleftrightarrow u(y_1, y_2) \geq u(z_1, z_2)$$