Nash Equilibrium in pure strategies

Nash Equilibrium

- Applying IDSDS:
  - Helps us delete all but one cell from the matrix in some games.
  - For other games, IDSDS deletes only a few strategies, providing a relatively imprecise equilibrium prediction.
  - And for other games, it does not have a bite.

Nash Equilibrium

- We next examine a different solution concept with "more bite", offering either the same or more precise equilibrium predictions.
- The "Nash Equilibrium", named after Nash (1950) builds on the notion that every player finds her "best response" to each of her rivals' strategies.

Best responses

- Best response. Player i regards strategy s_i as a best response to her rival's strategy s_j if s_i yields a weakly higher payoff than any other available strategy s_i' against s_j.

Best Response

- **Best response:**
  - A strategy s_i^* is a best response of player i to a strategy profile s_{-i} selected by all other players if it provides player i a larger payoff than any of his available strategies $s_i \in S_i$.

    $$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_i \in S_i$$

  - For two players, s_1^* is a best response to a strategy s_2 selected by player 2 if

    $$u_1(s_1^*, s_2) \geq u_1(s_1, s_2) \text{ for all } s_1 \in S_1$$

    That is, when player 2 selects s_2, the utility player 1 obtains from playing s_1^* is higher than by playing any other of his available strategies.

Nash Equilibrium

- Tool 12.2. *How to find best responses in matrix games*:
  1. Focus on the row player by fixing attention on one strategy of the column player.
     a) Cover with your hand all columns not being considered.
     b) Find the highest payoff for the row player by comparing the first component of every pair.
     c) Underline this payoff. This is the row player's best response to the column you considered from the column player.
  2. Repeat step 1, but fix your attention on a different column.
  3. For the column player, the method is analogous, but now direct your attention on one strategy of the row player.

Nash Equilibrium

- Nash equilibrium (NE). A strategy profile (s_i^*, s_j^*) is a NE if every player chooses a best response to her rivals' strategies.
    - A strategy profile is NE if it is a *mutual* best response: the strategy that player i chooses is a best response to that selected by player j, and vice versa.
    - As a result, no player has incentives to deviate because doing so would either lower her payoff, or keep it unchanged.

Nash Equilibrium

- Tool 12.3. *How to find Nash equilibria*:
  1. Find the best responses to all players.
  2. Identify which cell or cells in the matrix has all payoffs underlined, meaning that all players have a best response payoff. These cells are the NEs of the game.

Strategic Dominance

- *Example 12.4: Finding best responses and NEs.*
  - Consider matrix 12.7 (the same as in example 12.1):

		Firm 2 Tech a	Tech b
Firm 1	Tech A	$\underline{5}, 5$	$\underline{2}, 0$
	Tech B	3,2	1,1

Matrix 12.7

  - *Firm 1's best responses*.
    - When firm 2 chooses a, firm 1's best response is A because it yields a higher payoff than B, $5 > 3$.
    - When firm 2 chooses b, firm 1's best response is A, given that $2 > 1$.
    - Then, firm 1's best responses are $BR_1(a) = A$ when firm 2 chooses a and $BR_1(b) = A$, when firm 2 selects B.

Strategic Dominance

- *Example 12.4* (continued):

		Firm 2	
		Tech a	Tech b
Firm 1	Tech A	$\underline{5}, \underline{5}$	$\underline{2}, 0$
	Tech B	$3, \underline{2}$	$1,1$

Matrix 12.7

- *Firm 2's best responses*.
  - When firm 1 chooses A, $BR_2(A) = a$ because $5 > 0$.
  - When firm 1 chooses B, $BR_2(B) = a$ because $2 > 1$.
- *Faster tool: underling BR payoffs*.
  - The cells where all the payoffs are underlined must constitute a NE of the game because all players are playing mutual best responses.
- The NE is (A, a), the same prediction as IDSDS.

Strategic Dominance

- *Example 12.4* (continued):
  - Now consider matrix 12.8, which reproduces matrix 12.1b:

		Firm 2	
		Tech a	Tech b
Firm 1	Tech A	$\underline{5}, \underline{5}$	$\underline{2}, 0$
	Tech B	$3, \underline{1}$	$\underline{2}, \underline{1}$

Matrix 12.8

  - Firm 1's best responses are $BR_1(a) = A$ and $BR_1(b) = \{A, B\}$.
  - Firm 2's best responses are $BR_2(A) = a$ and $BR_2(B) = \{a, b\}$.
  - Strategy profiles (A, a) and (B, b) constitute the two NEs of the game.
  - The NE solution concept provides a more precise prediction than the IDSDS (which left with four strategies profiles).

Common Games

Common Games

- We apply the NE solution concept to 4 common games in economics and other social sciences:
  - The Prisoner's Dilemma game.
  - The Battle of the Sexes game.
  - The Coordination game.
  - The Anticoordination game.

Prisoner's Dilemma

- *Example 12.5: Prisoner's Dilemma game.*
  - Consider 2 people are arrested by the police, and are placed in different cells. They cannot communicate with each other.
  - The police have only minor evidence against them but suspects that the two committed a specific crime.
  - The police separately offers to each of them the deal represented in the following matrix (where negative values indicate disutility in years of jail):

		Player 2	
		Confess	Not confess
Player 1	Confess	$-5,-5$	$0,-10$
	Not confess	$-10,0$	$-1,-1$

Matrix 12.9a

Prisoner's Dilemma

- *Example 12.5* (continued):

		Player 2	
		Confess	Not confess
Player 1	Confess	$\underline{-5}, \underline{-5}$	$\underline{0}, -10$
	Not confess	$-10, \underline{0}$	$-1, -1$

Matrix 12.9a

- *Player 1's best responses* are:
  - $BR_1(C) = C$ because $-5 > -10$ and $BR_1(NC) = C$ because $0 > -1$.
- *Player 2's best responses* are:
  - $BR_2(C) = C$ because $-5 > -10$ and $BR_2(NC) = C$ because $0 > -1$.
- $(Confess, Confess)$ is the unique NE of the game, both players choose mutual best responses.

Prisoner's Dilemma

- In NE in the Prisoner's Dilemma game, every player, seeking to maximize her own payoff, confesses, which entails 5 years of jail for both.
- If instead, players could coordinate their actions and no confess, they would only serve 1 year in jail.
- This game illustrates strategic scenarios in which there is tension between individual incentives of each player and the collective interest of the group. *Examples*:
  - Price wars between firms.
  - Tariff wars between countries.
  - Use of negative campaigning in politics.

Battle of the Sexes

- *Example 12.6: Battle of the Sexes game.*
  - Ana and Felix are incommunicado in separate areas of the city.
  - In the morning, they talked about where to go after work, the football game or the opera, but they never agreed.
  - Each of them must simultaneously and independently choose where to go.
    - Ana and Felix's payoffs are symmetric. Each of them prefers to go to the event the other goes.

		Ana	
		Football	Opera
Felix	Football	5,4	3,3
	Opera	2,2	4,5

Matrix 12.10a

Battle of the Sexes

- *Example 12.6* (continued):

		Ana Football	Ana Opera
Felix	Football	$\underline{5}, \underline{4}$	3,3
	Opera	2,2	$\underline{4}, \underline{5}$

Matrix 12.10b

- *Felix's best responses* are:
  - $BR_{Felix}(F) = F$ because $5 > 2$ and $BR_{Felix}(O) = O$ because $4 > 3$.
- *Ana's best responses* are:
  - $BR_{Ana}(F) = F$ because $4 > 3$ and $BR_{Ana}(O) = O$ because $5 > 2$.
- The two NEs in this game are $(Football, Football)$ and $(Opera, Opera)$.

Coordination game

- *Example 12.7: Coordination game.*
  - Consider the game in matrix 12.11a illustrating a "bank run" between depositors 1 and 2, with payoffs in thousands of $.
  - News suggest that the bank where depositors 1 and 2 have their savings accounts could be in trouble.
  - Each depositor must decide simultaneously and independently whether to withdraw all the money in her account or wait.

		Depositor 2	
		Withdraw	Not withdraw
Depositor 1	Withdraw	50,50	100,0
	Not withdraw	0,100	150,150

Matrix 12.11a

Coordination game

- *Example 12.7* (continued):

		Depositor 2 Withdraw	Not withdraw
Depositor 1	Withdraw	$\underline{50}, \underline{50}$	$100{,}0$
	Not withdraw	$0{,}100$	$\underline{150}, \underline{150}$

Matrix 12.11b

- *Depositor 1's best responses* are:
  - $BR_1(W) = W$ because $50 > 0$ and $BR_1(NW) = NW$ because $150 > 100$.
- *Depositor 2's best responses* are:
  - $BR_2(W) = W$ because $50 > 0$ and $BR_2(NW) = NW$ because $150 > 100$.
- The two NEs in this game are $(Withdraw, Withdraw)$ and $(Not\ withdraw, Not\ witdraw)$.

Coordination game

- *Example 12.8: Anticoordination game.*
  - Matrix 12.12a presents a game with the opposite strategic incentives as the the Coordination game in example 12.7.
  - The matrix illustrates the Game of the Chicken, as seen in movies like *Rebel without a Cause* and *Footloose*.
  - Two teenagers in cars drive toward each other (or toward a cliff).
    - If the swerve they are regarded as "chicken."

		Player 2 Swerve	Stay
Player 1	Swerve	$-1, -1$	$-10, 10$
	Stay	$10, -10$	$-20, -20$

Matrix 12.12a

Coordination game

- *Example 12.8* (continued):

		Player 2	
		Swerve	Stay
Player 1	Swerve	$-1, -1$	$-10, 10$
	Stay	$10, -10$	$-20, -20$

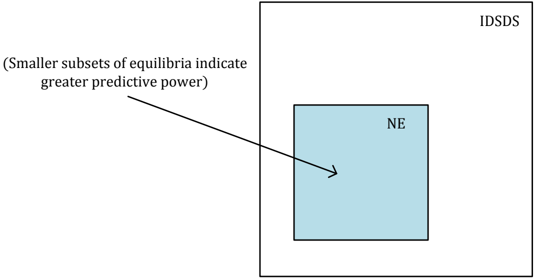
Matrix 12.12b

- *Player 1's best responses* are:
  - $BR_1(Swerve) = Stay$ because $10 > -1$ and $BR_1(Stay) = Swerve$ because $-10 > -20$.
- *Player 2's best responses* are:
  - $BR_2(Swerve) = Stay$ because $10 > -1$ and $BR_2(Stay) = Swerve$ because $-10 > -20$.
- The two NEs in this game are $(Swerve, Stay)$ and $(Stay, Swerve)$.

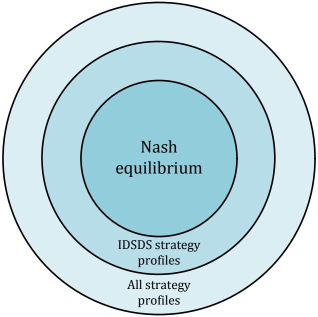
- Prisoner's Dilemma $\longrightarrow$ NE = set of strategies surviving IDSDS
- Battle of the Sexes $\longrightarrow$ NE is a subset of strategies surviving IDSDS (the entire game).

Therefore, NE has more predictive power than IDSDS.

- Great!



The NE provides more precise equilibrium predictions:



Hence, if a strategy profile (s_1^*, s_2^*) is a NE, it must survive IDSDS. However, if a strategy profile (s_1^*, s_2^*) survives IDSDS, it does not need to be a NE.

Rationalizable strategies

- Given the definition of a best response for player i, we can interpret that he will never use a strategy that cannot be rationalized for any beliefs about his opponents' strategies:
  - A strategy $s_i \in S_i$ is **never a best response** for player i if there are no beliefs he can sustain about the strategies that his opponents will select, s_{-i}, for which s_i is a best response.
  - We can then eliminate strategies that are never a best response from S_i, as they are not rationalizable.
- In fact, the only strategies that are rationalizable are those that survive such iterative deletion, as we define next:
  - A strategy profile $(s_1^*, s_2^*, ..., s_N^*)$ is **rationalizable** if it survives the iterative elimination of those strategies that are never a best response.
-

Rationalizable Strategies - Example

1. Beauty Contest / Guess the Average $[0, 100]$

The guess which is closest to $\frac{1}{2}$ the average wins a prize.

"Level 0"Players$\longrightarrow$They select a random number from $[0, 100]$, implying an average of 50.

$$
\begin{aligned}
\text{"Level 1" Players} &\longrightarrow BR(s_{-i}) = BR(50) = 25 \\
\text{"Level 2" Players} &\longrightarrow BR(s_{-1}) = BR(25) = 12.5 \\
\dots &\longrightarrow 0
\end{aligned}
$$

Rationalizable Strategies

How many degrees of iteration do subjects use in experimental settings?

- About 1-2 for "regular" people.
  - So they say $s_i = 50$ or $s_i = 25$.
- But...
  - One step more for undergrads who took game theory;
  - One step more for Portfolio managers;
  - 1-2 steps more for Caltech Econ majors;
  - About 3 more for usual readers of financial newspapers (*Expansión* in Spain and *FT* in the UK).

For more details, see Rosemarie Nagel "Unraveling in Guessing Games: An Experimental Study" (1995). *American Economic Review*, pp. 1313-26.

Some Questions about NE:

1. Existence? $\longrightarrow$ all the games analyzed in this course will have at least one NE (in pure or **mixed** strategies)
2. Uniqueness? $\longrightarrow$ Small predictive power. Later on we will learn how to restrict the set of NE.

Example 6: Rock-Paper-Scissors

- Not all games must have one NE using pure strategies...

		Lisa Rock	Paper	Scissors
	Rock	0, 0	-1, 1	1, -1
Bart	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

- **Bart's best responses:**
  - If Lisa chooses Rock, then Bart's best response is to choose Paper, i.e., $BR_B(Rock) = Paper$.
  - If Lisa chooses Paper, then Bart's best response is to choose Scissors, i.e., $BR_B(Paper) = Scissors$.
  - If Lisa chooses Scissors, then Bart's best response is to choose Rock, i.e., $BR_B(Scissors) = Rock$.

Example 6: Rock-Paper-Scissors

		Lisa Rock	Paper	Scissors
	Rock	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
Bart	Paper	$\underline{1}$, -1	0, 0	-1, $\underline{1}$
	Scissors	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- **Lisa's best responses:**
  - If Bart chooses Rock, then Lisa's best response is to choose Paper, i.e., $BR_L(Rock) = Paper$.
  - If Bart chooses Paper, then Lisa's best response is to choose Scissors, i.e., $BR_L(Paper) = Scissors$.
  - If Bart chooses Scissors, then Lisa's best response is to choose Rock, i.e., $BR_L(Scissors) = Rock$.

Example 6: Rock-Paper-Scissors

		Lisa Rock	Paper	Scissors
	Rock	0, 0	-1, <u>1</u>	<u>1</u>, -1
Bart	Paper	<u>1</u>, -1	0, 0	-1, <u>1</u>
	Scissors	-1, <u>1</u>	<u>1</u>, -1	0, 0

- In this game, there are no NE using pure strategies!
  - But it will have a NE using mixed strategies (In a couple of weeks).

Example 7: Game with Many Strategies

Player 2

Player 1		w	x	y	z
	a	0, 1	0, 1	1, 0	3, 2
	b	1, 2	2, 2	4, 0	0, 2
	c	2, 1	0, 1	1, 2	1, 0
	d	3, 0	1, 0	1, 1	3,1

- **Player 1's best responses:**
  - If Player 2 chooses w, then Player 1's best response is to choose d, i.e., $BR_1(w) = d$.
  - If Player 2 chooses x, then Player 1's best response is to choose b, i.e., $BR_1(x) = b$.
  - If Player 2 chooses y, then Player 1's best response is to choose b, i.e., $BR_1(y) = b$.
  - If Player 2 chooses z, then Player 1's best response is to choose a or d, i.e., $BR_1(z) = \{a, d\}$.

Example 7: Game with Many Strategies

Player 2 (columns), *Player 1* (rows)

	w	x	y	z
a	0, 1	0, 1	1, 0	3, 2
b	1, 2	2, 2	4, 0	0, 2
c	2, 1	0, 1	1, 2	1, 0
d	3, 0	1, 0	1, 1	3,1

- **Player 2's best responses:**
  - If Player 1 chooses a, then Player 2's best response is to choose z, i.e., $BR_1(a) = z$.
  - If Player 1 chooses b, then Player 2's best response is to choose w, x or z, i.e., $BR_1(b) = \{w, x, z\}$.
  - If Player 1 chooses c, then Player 2's best response is to choose y, i.e., $BR_1(c) = y$.
  - If Player 1 chooses d, then Player 2's best response is to choose y or z, i.e., $BR_1(d) = \{y, z\}$.

Example 7: Game with Many Strategies

Player 2 (columns), *Player 1* (rows)

	w	x	y	z
a	0, 1	0, 1	1, 0	$\underline{3}$, $\underline{2}$
b	1, $\underline{2}$	$\underline{2}$, $\underline{2}$	$\underline{4}$, 0	0, $\underline{2}$
c	2, 1	0, 1	1, $\underline{2}$	1, 0
d	$\underline{3}$, 0	1, 0	1, $\underline{1}$	$\underline{3}$,$\underline{1}$

- NE can be applied very easily to games with many strategies. In this case, there are 3 seperate NE: (b,x), (a,z) and (d,z).
- Two important points:
  - Note that BR cannot be empty: I might be indifferent among my available strategies, but BR is non-empty.
  - Another important point: Players can use weakly dominated strategies, i.e., a or d by Player 1; y or z by Player 2.

Example 8: The American Idol Fandom

- We can also find the NE in 3-player games.
  - Harrington, pp. 101-102.
  - More generally representing a coordination game between three individuals or firms.
- "Alicia, Kaitlyn, and Lauren are ecstatic. They've just landed tickets to attend this week's segment of American Idol. The three teens have the same favorite among the nine contestants that remain: Ace Young. They're determined to take this opportunity to make a statement. While [text]ing, they come up with a plan to wear T-shirts that spell out "ACE" in large letters. Lauren is to wear a T-shirt with a big "A," Kaitlyn with a "C," and Alicia with an "E." If they pull this stunt off, who knows–they might end up on national television! OMG!

Example 8: The American Idol Fandom

- While they all like this idea, each is tempted to wear instead an attractive new top just purchased from their latest shopping expedition to Bebe. It's now an hour before they have to leave to meet at the studio, and each is at home trying to decide between the Bebe top and the lettered T-shirt. What should each wear?"

Alicia chooses E

		Kaitlyn: C	Kaitlyn: Bebe
Lauren	A	2, 2, 2	0, 1, 0
	Bebe	1, 0, 0	1, 1, 0

Alicia chooses Bebe

		Kaitlyn: C	Kaitlyn: Bebe
Lauren	A	0, 0, 1	0, 1, 1
	Bebe	1, 0, 1	1, 1, 1

Example 8: The American Idol Fandom

Alicia chooses E

		Kaitlyn C	*Kaitlyn* Bebe
Lauren	A	$\underline{2}, \underline{2}, \underline{2}$	$0, 1, 0$
Lauren	Bebe	$1, 0, 0$	$\underline{1}, \underline{1}, 0$

Alicia chooses Bebe

		Kaitlyn C	*Kaitlyn* Bebe
Lauren	A	$0, 0, 1$	$0, \underline{1}, \underline{1}$
Lauren	Bebe	$\underline{1}, 0, \underline{1}$	$\underline{1}, \underline{1}, \underline{1}$

- There are 2 psNE: (A,C,E) and (Bebe, Bebe, Bebe)

Example 9: Voting: Sincere or Devious?

- Harrington pp. 102-106
- Three shareholders (1, 2, 3) must vote for three options (A, B, C) where
  - Shareholder 1 controls 25% of the shares
  - Shareholder 2 controls 35% of the shares
  - Shareholder 3 controls 40% of the shares
- Their preferences are as follows:

Shareholder	1st Choice	2nd Choice	3rd Choice
1	A	B	C
2	B	C	A
3	C	B	A

Example 9: Voting: Sincere or Devious?

3 votes for A

Player 1 (rows) / Player 2 (columns)

1 \ 2	A	B	C
A	A	A	A
B	A	B	A
C	A	A	C

3 votes for B

Player 1 (rows) / Player 2 (columns)

1 \ 2	A	B	C
A	A	B	B
B	B	B	B
C	B	B	C

3 votes for C

Player 1 (rows) / Player 2 (columns)

1 \ 2	A	B	C
A	A	C	C
B	C	B	C
C	C	C	C

This implies the following winners, for each possible strategy profile:

Example:
1 votes B, 2 votes B, 3 votes C:
Votes for B = 25 + 35 = 60%
Votes for C = 40%

B is the Winner

Example 9: Voting: Sincere or Devious?

3 votes for A

Player 1 (rows) vs. Player 2 (columns)

	A	B	C
A	2,0,0	2,0,0	2,0,0
B	2,0,0	1,2,1	2,0,0
C	2,0,0	2,0,0	0,1,2

3 votes for B

Player 1 (rows) vs. Player 2 (columns)

	A	B	C
A	2,0,0	1,2,1	1,2,1
B	1,2,1	1,2,1	1,2,1
C	1,2,1	1,2,1	0,1,2

3 votes for C

Player 1 (rows) vs. Player 2 (columns)

	A	B	C
A	2,0,0	0,1,2	0,1,2
B	0,1,2	1,2,1	0,1,2
C	0,1,2	0,1,2	0,1,2

Each player obtains a payoff of:
2 if his most preferred option is adopted
1 if his second most preferred option is adopted
0 if his least preferred option is adopted

Example 9: Voting: Sincere or Devious?

3 votes for A

1 \ 2

	A	B	C
A	2,0,0	2,0,0	2,0,0
B	2,0,0	1,2,1	2,0,0
C	2,0,0	2,0,0	0,1,2

3 votes for B

1 \ 2

	A	B	C
A	2,0,0	1,2,1	1,2,1
B	1,2,1	1,2,1	1,2,1
C	1,2,1	1,2,1	0,1,2

3 votes for C

1 \ 2

	A	B	C
A	2,0,0	0,1,2	0,1,2
B	0,1,2	1,2,1	0,1,2
C	0,1,2	0,1,2	0,1,2

5 NEs:
(A, A, A)
(B, B, B)
(C, C, C)
(B, B, C)
(A, C, C)

A comment on the NEs we just found

- **First point** :Sincere voting cannot be supported as a NE of the game.
    - Indeed, for sincere voting to occur, we need that each player selects his/her most preferred option, i.e., profile (A,B,C), which is not a NE.
- **Second point:** In the symmetric strategy profiles (A,A,A), (B,B,B), and (C,C,C), no player is pivotal, since the outcome of the election does not change if he/she were to vote for a different option.
    - That is, a player's equilibrium action is weakly dominant.

A comment on the NEs we just found

3 votes for A

2

1		A	B	C
	A	A	A	A
1	B	A	B	A
	C	A	A	C

This result can be easily visualized by analyzing the matrices representing the results of each voting profile.

For instance, for (A,A,A), option A arises. If only one player changes his vote, option A is still the winner.

3 votes for B

2

		A	B	C
	A	A	B	B
1	B	B	B	B
	C	B	B	C

A similar argument is applicable to (B,B,B) ...

3 votes for C

2

		A	B	C
	A	A	C	C
1	B	C	B	C
	C	C	C	C

... and to (C,C,C).

A comment on the NEs we just found

- **Third point:** Similarly, in equilibrium (B,B,C), shareholder 3 does not have incentives to deviate to a vote different than C since he would not be able to change the outcome.
    - Similarly for shareholder 1 in equilibrium (A,C,C).

A comment on the NEs we just found

3 votes for A

1 \ 2	A	B	C
A	A	A	A
B	A	B	A
C	A	A	C

3 votes for B

1 \ 2	A	B	C
A	A	B	B
B	B	B	B
C	B	B	C

3 votes for C

1 \ 2	A	B	C
A	A	C	C
B	C	B	C
C	C	C	C

In NE (B,B,C), option B is the winner.

In (B,B,C) a unilateral deviation of player 3 towards voting for A (in the top matrix) or for B (in the middle matrix) ***still*** yields option B as the winner. Player 3 therefore has no incentives to unilaterally change his vote.

Cournot Model

- Consider an industry with $N = 2$ firms selling a homogeneous product.
- Every firm independently and simultaneously chooses its profit maximizing output (q_1 for firm 1 and q_2 for firm 2).
- The market price is determined by inserting q_1 and q_2 into the inverse demand function $p(q_1, q_2)$. Assume this function is linear, $p(q_1, q_2) = a - b(q_1+q_2)$, where $a, b > 0$.
- Firm 1's total cost function is $TC_1(q_1)=cq_1$, where $c > 0$.
- Firm 2's total cost function is symmetric, $TC_2(q_2) = cq_2$.

Cournot Model

Firm 1. Its PMP is to choose q_1 to solve

$$\max_{q_1} \pi_1 = TR_1 - TC_1 = \underbrace{p(q_1, q_2)q_1}_{TR_1} - \underbrace{cq_1}_{TC_1}$$

$$= [a - b(q_1 + q_2)]q_1 - cq_1,$$

where $TR_1 = p(q_1, q_2)q_1$ denotes total revenue (price per units sold), and $TC_1 = cq_1$ is its total cost.

- To maximize its profits, firm 1 differentiate this expression with respect to q_1,

$$\frac{\partial \pi_1}{q_1} = a - 2bq_1 - bq_2 - c = 0.$$

Rearranging and solving for q_1,

Cournot Model

$$a - c - bq_2 = 2bq_1,$$

$$q_1(q_2) = \frac{a-c}{2b} - \frac{1}{2}q_2, \qquad (BRF_1)$$

which is referred to as firm 1's "best response function."

- The best response function describes the profit maximizing output that firm 1 chooses as a response to each of the output levels that firm 2 selects.
  - If $a = 10$, $b = 1$, and $c = 2$, firm 1's best response function becomes

$$q_1(q_2) = \frac{10-2}{2 \times 1} - \frac{1}{2}q_2 = 4 - \frac{1}{2}q_2.$$

  - If firm 2 produces $q_2 = 3$ units, firm 1 responds with

$$q_1(2) = 4 - \frac{1}{2}2 = 2.5 \text{ units.}$$

Cournot Model

- Firm 1's best response function, $q_1(q_2) = \frac{a-c}{2b} - \frac{1}{2} q_2$.
  - It originates at $\frac{a-c}{2b}$ on the vertical axis when firm 2 chooses $q_2 = 0$.
  - It decreases with a slope of $-1/2$ for every unit of q_2.
  - When $q_1\left(\frac{a-c}{b}\right) = \frac{a-c}{2b} - \frac{1}{2}\frac{a-c}{2b}$
    $= 0$ units.

  As firm 2 increases q_2, firm 1 is left with a smaller residual demand to serve.

  When $q_2 \geq \frac{a-c}{b}$, firm 1 shut down, producing $q_1 = 0$.

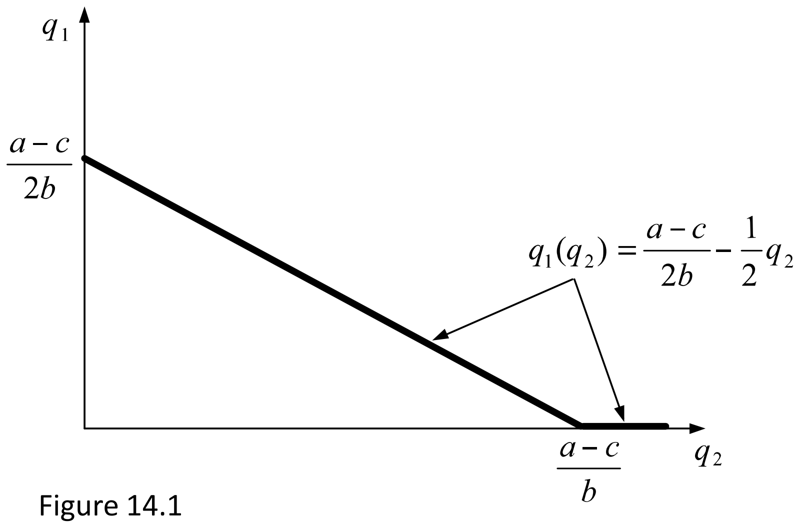


Figure 14.1

Cournot Model

Firm 2. A similar argument applies to firm 2, which solves

$$\max_{q_2} \pi_2 = TR_2 - TC_2 = \underbrace{p(q_1, q_2)q_2}_{TR_2} - \underbrace{cq_2}_{TC_2}$$

$$= [a - b(q_1 + q_2)]q_2 - cq_2.$$

- Differentiating with respect to q_2,

$$\frac{\partial \pi_2}{q_2} = a - bq_1 - 2bq_2 - c = 0.$$

Rearranging and solving for q_2, we find firm 2's best response function,

$$a - c - bq_1 = 2bq_2,$$

$$q_2(q_1) = \frac{a - c}{2b} - \frac{1}{2}q_1. \qquad (BRF_2)$$

Cournot Model

- Firm 2's best response function, $q_2(q_1) = \frac{a-c}{2b} - \frac{1}{2}q_1$, is symmetric to that of firm 1 because both face the same demand and costs.
  - It originates at $\frac{a-c}{2b}$ when firm 1 is inactive but it decreases at a rate of $1/2$ as firm 1 increases its production.

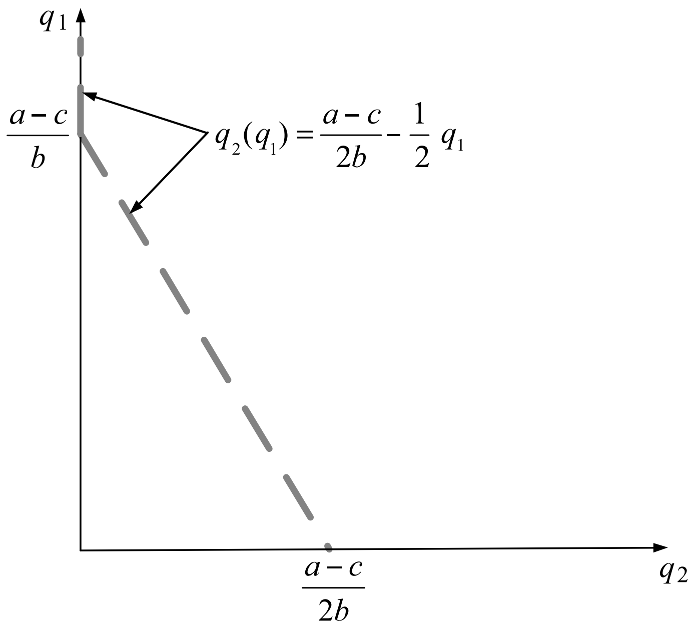


Figure 14.2

Cournot Model

- Superimposing firm 1's and firm 2's best response functions, we obtain their crossing point: Cournot Equilibrium.

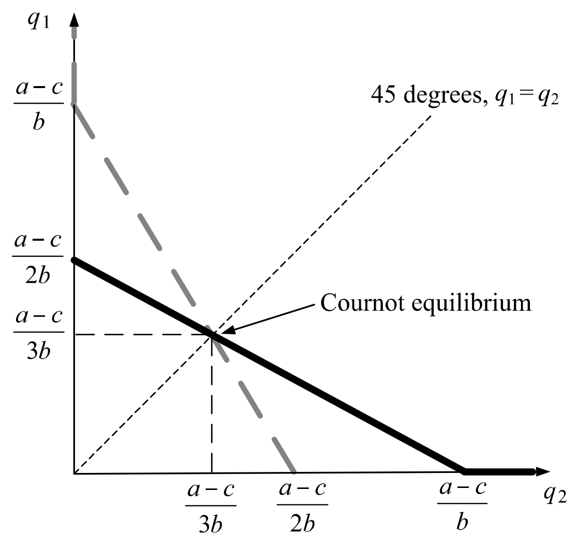


Figure 14.3

- Both firms are choosing output levels that are the best response to the output of its rival (i.e., firms are selecting *mutual* best responses, which is the Nash Equilibrium (NE) of a game).

Cournot Model

- To find the point where the best response functions cross each other, we can insert BRF_2 into BRF_1,

$$q_1 = \frac{a-c}{2b} - \frac{1}{2}\underbrace{\left(\frac{a-c}{2b} - \frac{1}{2}q_1\right)}_{q_2},$$

- Rearranging and solving for q_1, we find q_1^*,

$$\frac{3}{4}q_1 = \frac{a-c}{2b},$$

$$q_1^* = \frac{a-c}{3b}.$$

Cournot Model

- Inserting this output level into BRF_1, we find q_2^*,

$$q_2\left(\frac{a-c}{3b}\right) = \overbrace{\frac{a-c}{2b}}^{q_1^*} - \frac{1}{2}\frac{a-c}{3b}$$

$$= \frac{3(a-c)-(a-c)}{6b},$$

$$q_2^* = \frac{a-c}{3b}.$$

Cournot Model

- The output pair $(q_1^*, q_2^*) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$ is the Nash Equilibrium of the Cournot game.

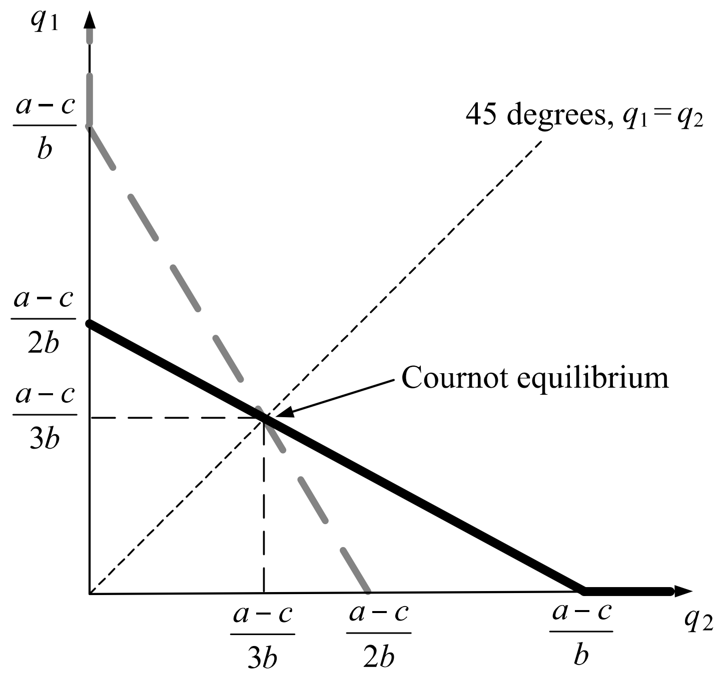


Figure 14.3

Cournot Model

- An alternative approach to solve for the equilibrium output is to invoke symmetry.
- Because firms are symmetric in their revenues and costs, we can claim that there must be a symmetric equilibrium where

$$q_1^* = q_2^* = q^*.$$

- Inserting this property into either firm's BRF,

$$q^* = \frac{a-c}{2b} - \frac{1}{2}q^*,$$
$$\frac{3}{2}q^* = \frac{a-c}{2b},$$
$$q^* = \frac{a-c}{3b}.$$

Cournot Model

- We find equilibrium price by evaluating the inverse demand function

$$p(q_1, q_2) = a - b(q_1 + q_2)$$

at $q_1^* = q_2^* = \frac{a-c}{3b}$,

$$p^*\left(\frac{a-c}{3b}, \frac{a-c}{3b}\right) = a - b\left(\frac{a-c}{3b} + \frac{a-c}{3b}\right)$$

$$= a - \frac{2(a-c)}{3} = \frac{a+2c}{3}.$$

Cournot Model

- Finally, equilibrium profits for every firm $i = \{1,2\}$ are

$$\pi_i^* = p^* q_i^* - c q_i^* = \left(\frac{a+2c}{3}\right)\frac{a-c}{3b} - c\,\frac{a-c}{3b}$$

$$= \frac{(a+2c)(a-c)}{9b} - \frac{3c(a-c)}{9b}$$

$$= \frac{a^2 - 2ac + c^2}{9b},$$

or, more compactly,

$$\pi_i^* = \frac{(a-c)^2}{9b}$$

because $(a-c)^2 = a^2 - 2ac - c^2$.

It can be alternatively expressed as $\pi_i^* = (q^*)^2$.

Cournot Model

- *Example 14.1: Cournot model with symmetric costs.*
  - Consider a duopoly with $p(q_1, q_2) = 12 - q_1 - q_2$, where every firm $i = \{1,2\}$ faces a symmetric cost function $TC_i(q_i) = 4q_i$.
  - *Firm 1's best response function.* Firm 1 chooses q_1 to solve

$$\max_{q_1} \pi_1 = (12 - q_1 - q_2)q_1 - 4q_1.$$

Differentiating with respect to q_1,

$$\frac{\partial \pi_1}{q_1} = 12 - 2q_1 - q_2 - 4 = 0.$$

Rearranging and solving for q_1,

$$8 - q_2 = 2q_1,$$

$$q_1(q_2) = 4 - \frac{1}{2}q_2. \qquad (BRF_1)$$

Cournot Model

- *Example 14.1* (continued):
  - *Firm 2's best response function.* Firm 2 chooses q_2 to solve

$$\max_{q_2} \pi_2 = (12 - q_1 - q_2)q_2 - 4q_2.$$

Differentiating with respect to q_2,

$$\frac{\partial \pi_2}{q_2} = 12 - q_1 - 2q_2 - 4 = 0.$$

Rearranging and solving for q_1,

$$8 - q_1 = 2q_2,$$

$$q_2(q_1) = 4 - \frac{1}{2}q_1, \qquad (BRF_2)$$

which is symmetric to that of firm 1.

Cournot Model

- *Example 14.1* (continued):
  - *Finding equilibrium output.*

    We can invoke symmetry, and claim

$$q_1^* = q_2^* = q^*.$$

Inserting this property into either firm's best response function, and solving for q^*,

$$q^* = 4 - \frac{1}{2}q^*,$$

$$\frac{3}{2}q^* = 4 \quad \Longrightarrow q^* = \frac{8}{3}.$$

Cournot Model

- *Example 14.1* (continued):
  - *Finding equilibrium output* (cont.).

    Equilibrium price is

    $$p^*\left(\frac{8}{3},\frac{8}{3}\right) = 12 - q^* - q^* = 12 - \frac{8}{3} - \frac{8}{3} = \frac{20}{3} \cong \$6.67,$$

    producing for every firm $i = \{1,2\}$ equilibrium profits of

    $$\pi_i^* = p^*q^* - cq^* = \left(\frac{20}{3}\right)\frac{8}{3} - 4\frac{8}{3} = \frac{160}{9} - \frac{96}{9} = \frac{64}{9}.$$

Bertrand Model– Simultaneous Price Competition

Bertrand Model

- Two symmetric firms produce a homogeneous good and face common marginal cost, $c > 0$.
- They simultaneously and indecently set prices p_1 and p_2.
  - If $p_1 < p_2$, firm 1 captures all the demand, while firm 2 captures none: $$x_1(p_1, p_2, I) > 0,$$ $$x_2(p_1, p_2, I) = 0.$$
  - If $p_1 > p_2$, firm 2 captures all demand.
  - If $p_1 = p_2$, both firms equally share market demand: $$\frac{1}{2}x_1(p_1, p_2, I) > 0,$$ $$\frac{1}{2}x_2(p_1, p_2, I) > 0.$$

Bertrand Model

- The Bertrand model of price competition claims that, in equilibrium:

$$p_1 = p_2 = c.$$

- To show this result, we next demonstrate that all possible price pairs (p_1, p_2) that are different from $(p_1, p_2) = (c, c)$, cannot be equilibria.

Bertrand Model

- We need to show that any price different than the marginal cost, c, is “unstable” in the sense that at least one firm has incentives to deviate to a different price.
- We examine:
  1. Asymmetric price profiles, where $p_1 \neq p_2$.
  2. Symmetric price profiles, where $p_1 = p_2$.

Bertrand Model

1. *Asymmetric price profiles*.

   (a) Consider $p_1 > p_2 > c$.

   - Firm 2 sets the lowest price and captures the entire market by making a positive margin because $p_2 > c$.
   - This profile cannot be stable because firm 1 has incentives to deviate undercutting firm2's prices by charging $p_1' = p_2 - \varepsilon$, where ε indicates a small reduction in firm 2's price.

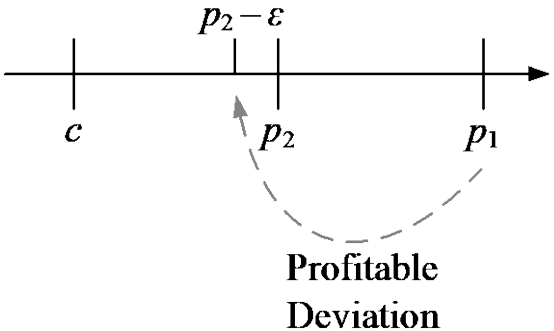


Figure 14.4

Bertrand Model

1. *Asymmetric price profiles* (cont.).

   (b) Consider $p_1 > p_2 = c$.

   - Firm 2 captures the entire market, but it makes no profit per unit.
   - Firm 1 would not have incentives to undercut firm 2's price that would entail charging a price below c, incurring in a per unit cost.
   - Instead, firm 2 would have incentives to deviate by increasing its price from $p_2 = c$ to slightly below its rival's price, $p_2' = p_1 - \varepsilon$, and make a higher profit.

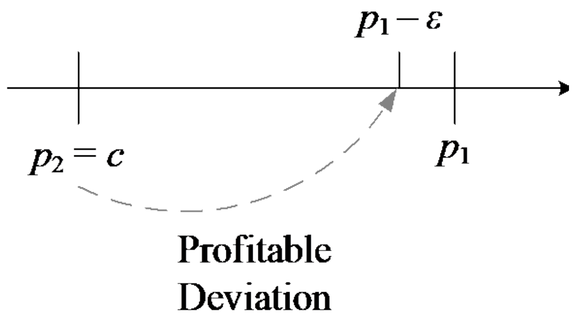


Figure 14.5

Bertrand Model

2. *Symmetric price profiles*.

(a) Consider $p_1 = p_2 > c$.

- Both firms evenly share the market because their prices are the same.
- Evert firm i has the incentive to deviate by undercutting its rival's price p by a small amount ε, by charging $p_i' = p - \varepsilon$.

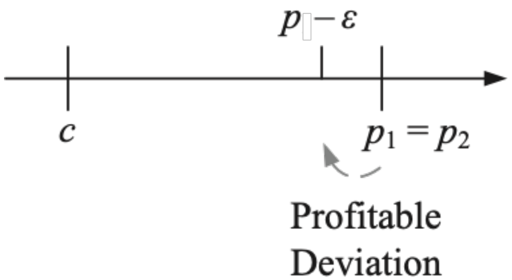


Figure 14.6

Bertrand Model

2. *Symmetric price profiles (cont.).*

(b) Consider $p_1 = p_2 = c$.

- Prices coincide, leading firms to evenly share the market.
- These prices leave no positive margin per unit because $p_i = c$ for every firm i.
- No firm can strictly increase its payoff by unilaterally deviating:
  - A lower price would attract all consumers, but at a lower per unit loss.
  - A higher price would reduce the deviating firm's sales to zero.

We can claim that setting $p_i = c$ is a weakly dominant strategy in the Bertrand model of price competition because no firm can strictly increase its profit by deviating from such a price.

Bertrand Model

- *Example 14.3: Bertrand model.*
  - Consider the inverse demand function in example 14.1, $p(q_1, q_2) = 12 - q_1 - q_2$.
  - Because $Q \equiv q_1 + q_2$ denotes the aggregate output in the industry, the inverse demand can be expressed as

$$p(Q) = 12 - Q.$$

  - In the Bertrand model of price competition, all firms in the industry lower their prices until

$$p = c \quad \Longrightarrow 12 - Q = c.$$

  Solving for Q, $Q = 12 - c$.
    - If $c = 4$, $Q = 12 - 4 = 8$ units, each of which sold at a price of $\$4$.

Reconciling the Cournot and Bertrand models

- *Why are the results in the Cournot model and Bertrand model so dramatically different?*
  - In the Cournot model,
    - firms set a price above marginal cost, making a positive profit.
  - In the Bertrand model,
    - firms set $p = c$, earning no economic profits.

Reconciling the Cournot and Bertrand models

- These differences are driven by the absence of capacity constraints in the Bertrand model:
  - If a firm charges 1 cent less than its rival, it captures the market demand, regardless of its size.
- This assumption might be reasonable for goods such as online movie streaming
  - but difficult to justify with others (e.g., smartphones) with a world demand that cannot be served by a single firm.