"Split or Steal" in Golden Balls

- Golden Ball was a British TV game show during 2007 - 2009. At the last phase of the game - known as "Split or Steal", a procedure to allocate a certain amount of money between the players is used
  - Both players simultaneously choose a ball placed in front of them. Inside, one is printed with "Split" and the other one is printed with "Steal"
  - If both players choose *Split*, they divide the money; if one chooses *Split* while the other chooses *Steal*, the one who chooses *Steal* gets all the money while the other gets nothing; if both choose *Steal*, each one gets nothing
  - An example is illustrated in the following table

		Nick	
		Split	*Steal*
Abraham	*Split*	6800, 6800	0, 13600
	Steal	13600, 0	0, 0

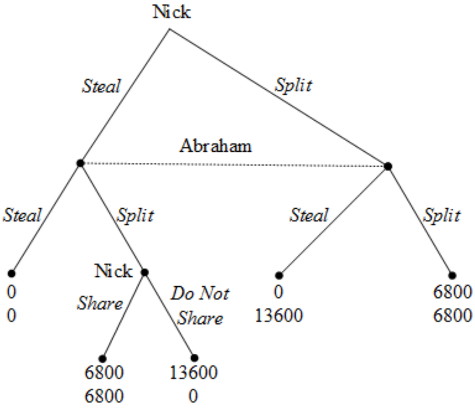
"Split or Steal" in Golden Balls

- You can verify that *Steal* weakly dominates *Split* for both Abraham and Nick.
- We can also find 3 pure strategy Nash equilibria in this game, namely (*Steal*, *Split*), (*Split*, *Steal*), and (*Steal*, *Steal*).
- The following matrix underlines best response payoffs:

		Nick: *Split*	Nick: *Steal*
Abraham	*Split*	6800, 6800	<u>0</u>, <u>13600</u>
	Steal	<u>13600</u>, <u>0</u>	<u>0</u>, <u>0</u>

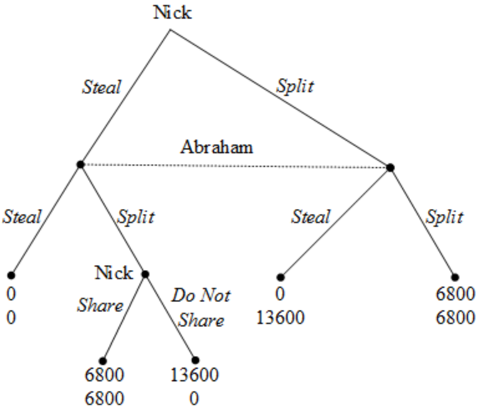
"Split or Steal" in Golden Balls

- Nick comes up with a surprising proposal:
  - "Abraham, I want you to trust me 100%. I'm going to pick the steal ball. I want you to do split and I promise you that I will split the money with you."



"Split or Steal" in Golden Balls

- Abraham is taken aback and attempts to convince Nick that they should both choose the split ball. After referring to Nick as an idiot whose plan would yield nothing to both of them, Nick finally consents.
- As a result, they both choose *Split* and divide the money.



"Split or Steal" in Golden Balls

- One possible explanation is that Nick was trying to alter the situation in the eyes of Abraham...
  - so that *Steal* no longer weakly dominates *Split*, as illustrated in the following matrix.

		Nick			
		Split/ Share	*Split/ Do not share*	*Steal/ Share*	*Steal/ Do not share*
Abraham	*Split*	6800, 6800	6800, 6800	6800, 6800	0, 13600
	Steal	13600, 0	13600, 0	0, 0	0, 0

"Split or Steal" in Golden Balls

- As you can see from the strategy pairs, *Steal* no longer dominates *Split* for Abraham because:
  - when Nick chooses *Steal/Share*, Abraham is strictly better off when he chooses *Split* than when he chooses *Steal.*
- But this is not a strictly dominating strategy because for Nick as *Steal/Do not share* strictly dominates *Steal/Share.*
- Nick's plan was not to steal and share the proceeds.
  - Instead to give Abraham a reason for choosing *Split*...
  - so that Nick could feel comfortable in choosing *Split*

Bargaining games

Bargaining Games

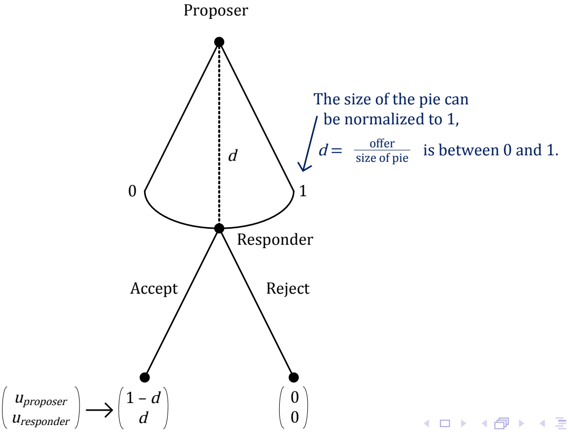
- Bargaining is prevalent in many economic situations where two or more parties negotiate how to divide a certain surplus.
- These strategic settings can be described as a sequential-move game where one player is the first mover in the game, proposing a certain "split of the pie" among all players.
    - The players who receive the offer must then choose whether to accept the offer or reject it (considering that, in such case, they might have the opportunity to make counteroffers).

Bargaining Games

- Let's start with a simple bargaining game in which counteroffers are not allowed.
  - This is the so-called "Ultimatum Bargaining" game.
- We will then examine more elaborate bargaining games, where receives can make counteroffers.
  - Afterwards, we will even allow the initial proposer to make a counter-counteroffer, etc.
- Difficult? No!
  - We will be using backward induction in all these examples to find the SPNE.

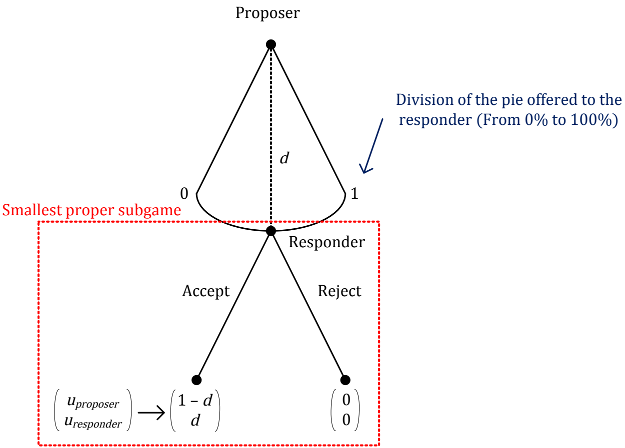
Ultimatum bargaining game

- Take-it-or-leave-it offer:
  - The proposer makes an offer d to the responder, and if the offer is accepted, the proposer keeps the remaining pie, $1 - d$.



Bargaining Games

- Applying backward induction in the ultimatum bargaining game



Ultimatum bargaining game

- Let us use backward induction:
  - First, the responder accepts any offer such that

$$u_R(Accept) \geq u_R(Reject) \iff d \geq 0$$

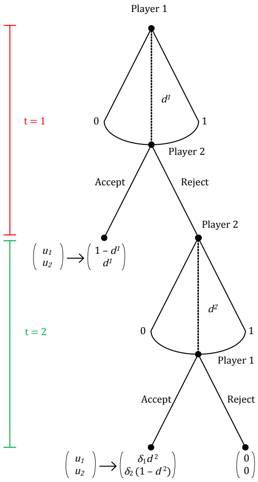
  - Second, the proposer, anticiparing that any offer $d \geq 0$ is accepted by the responder, he chooses the level of d that maximizes his utility (his utility is the remaining pie, $1 - d$). That is,

$$\max_{d \geq 0} \; 1 - d$$

- Taking FOCs with respect to d yields -1 (corner solution), so the optimal division is d*=0

- Therefore, the SPNE of the game prescribes that:
  - The proposer makes an offer $d^* = 0$; and
  - The responder accepts any offer $d \geq 0$.
- Note that we don't say something as restrictive as:
  - "The responder accepts the equilibrium offer of the proposer $d^* = 0$,". Instead, we describe what he would do (Accept/Reject) after receiving any offer d from the proposer.
  - This is a common property when describing the SPNE of a game in order to account for both in-equillibrium and off-the-equillibrium behavior.

Two-period alternating offers bargaining game



Two-period alternating offers bargaining game

Using backward induction:
During period $t = 2$,
Player 1 accepts any offer d^2 coming from player 2 iff $\delta_1 d^2 \geq 0$, i.e., $d^2 \geq 0$.
Player 2, knowing that player 1 accepts any offer d^2 satisfying $d^2 \geq 0$, makes an offer maximizing his utility function

$$\underbrace{\max_{d^2 \geq 0} \delta_2 \left(1 - d^2\right) \Longrightarrow d^2 = 0}_{\text{Analog to the Ultimatum Bargaining Game}}$$

which gives her a payoff of $\delta_2 (1 - 0) = \delta_2$.

During period $t = 1$,
Player 2 rejects any offer d^1 from player 1 that is below what she will get for herself during the next period, δ_2, i.e., she rejects any offer d^1 such that

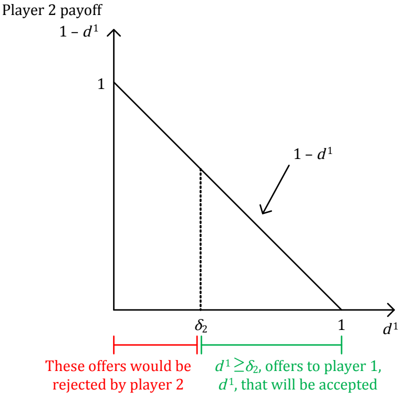
$$\delta_2 > d^1$$

Player 1 then offers to player 2 an offer d^1 such that maximizes his own utility, and guarantees that player 2 accepts such offer (i.e., $d^1 > \delta_2$), that is,

$$\max_{d^1 \geq \delta_2} 1 - d^1 \Longrightarrow d^1 = \delta_2$$

which gives him a payoff of $1 - \delta_2$.

Two-period alternating offers bargaining game



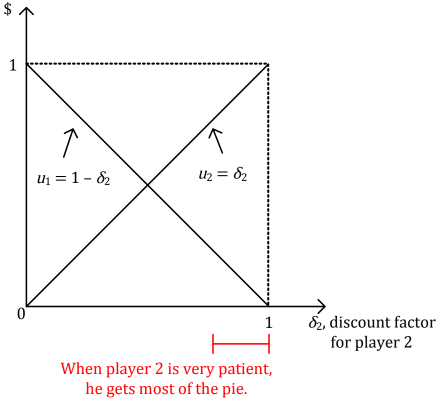
- Among all offers to player 1 that will be accepted, $d^1 \geq \delta_2$, the offer $d^1 = \delta_2$ provides player 2 the highest expected possible payoff.

- Therefore, we can describe the SPNE of this game as follows:
  - **Player 1** offers $d^1 = \delta_2$ in period t=1, and accepts any offer $d^2 \geq 0$ in t=2; and
  - **Player 2** offers $d^2 = 0$ in period t=2, and accepts any offer $d^1 \geq \delta_2$ in t=1.

- As a consequence, the SPNE payoffs are $(1 - \delta_2, \delta_2)$, and the game ends at the first stage.
- Note also that, the more patient player 2 is (higher δ_2), the more he gets and the less player 1 gets in the SPNE of the game.
  - (Figure on next slide)

Two-period alternating offers bargaining game

- Equilibrium payoffs for player 1 and 2 in the two-period alternating offers bargaining game



- Here we saw a very useful trick to solve longer alternating o ffer bargaining games.
    - (Figures in the next slides:the "ladder method")

Two-period alternating offers bargaining game

- A useful trick for alternating offers bargaining games:

Proposing Player	Time Period	
P_1	$t = 1$	$(1 - \delta_2, \overbrace{\delta_2 \cdot 1}^{\longleftarrow \text{Remaining}})$
		$\uparrow$
P_2	$t = 2$	$(0, \overbrace{1})$

- Player 2 is indifferent between offering himself the entire pie in period $t = 2$, or receiving in period $t = 1$ an offer from player 1 equal to the discounted value of the entire pie.
- SPNE:

$$P_1 \begin{cases} \text{Offers } d^1 = \delta_2 \text{ in period 1, and} \\ \text{Accepts any offer } d^2 \geq 0 \text{ in period 2.} \end{cases}$$

$$P_2 \begin{cases} \text{Offers } d^2 = 0 \text{ in period 2, and} \\ \text{Accepts any offer } d^1 \geq \delta_2 \text{ in period 1.} \end{cases}$$

Four-period alternating offers bargaining game

- Generalizing this trick to more periods:

Proposer	Period	
P_2	$t = 1$	$(\overbrace{\delta_1(1-\delta_2(1-\delta_1))}^{\text{Remaining} \longrightarrow}, 1-\delta_1(1-\delta_2(1-\delta_1)))$
		$\uparrow \qquad \longleftarrow\text{Remaining}$
P_1	$t = 2$	$(\overbrace{1-\delta_2(1-\delta_1)}, \overbrace{\delta_2(1-\delta_1)})$
		$\text{Remaining} \longrightarrow \qquad \uparrow$
P_2	$t = 3$	$(\overbrace{\delta_1 \cdot 1}, \overbrace{1-\delta_1})$
		$\uparrow$
P_1	$t = 4$	$(\overbrace{1}, 0)$

- Pretty fast when dealing with multiple periods!

Bargaining over infinite periods

- Watson, pp. 220-222
- Remember the **finite** bargaining models we just covered?
- In those models, we allowed players to bargain over a surplus (or a pie) for a finite number of periods, $T = 2, 3, 4, \ldots$
- What if we allow them to negotiate for as many periods as they need?
- (Of course they would not bargain forever, since they discount the future.)

Bargaining over infinite periods

- Player 1 makes offers to player 2, d_2, in periods $1, 3, 5, \ldots$
- Player 2 makes offers to player 1, d_1, in periods $2, 4, 6, \ldots$
- Hence, at every *odd* period, player 2 compares the payoff he gets by accepting the offer he receives from player 1, d_2, with respect to...
  - the payoff he can get tomorrow by making an offer of d_1 to player 1, and keeping the rest of the pie for himself, $1 - d_1$
  - In addition, this payoff is discounted, since it is received tomorrow. Hence, $\delta_2(1 - d_1)$.
- Therefore, player 2 accepts the offer d_2 from player 1 if and only if:

$$d_2 \geq \delta_2(1 - d_1)$$

Bargaining over infinite periods

- And since player 1 wants to minimize the offer he makes to player 2, d_2, in order to keep the largest remaining pie for himself, player 1 will offer the minimal division to player 2, d_2, that guarantees acceptance

$$d_2 \overbrace{=}^{\geq} \delta_2 (1 - d_1)$$

  - Importantly, this is valid at every *odd* period $t = 1, 3, 5, \ldots$ (not only at period $t = 1$),

Bargaining over infinite periods

- Similarly, at every *even* period $t = 2, 4, 6, \ldots$, player 1 compares the payoff he gets by accepting the offer he receives, d_1, with respect to...
  - the payoff he can get tomorrow by making an offer of d_2 to player 2, and keeping the rest of the pie for himself, $1 - d_2$
  - In addition, this payoff is discounted. since it is received tomorrow. Hence, $\delta_1(1 - d_2)$.
- Therefore, player 1 accepts the offer d_1 from player 2 if and only if:

$$d_1 \geq \delta_1(1 - d_2)$$

Bargaining over infinite periods

- And since player 2 wants to minimize the offer he makes to player 1, d_1, in order to keep the largest remaining pie for himself, player 2 will offer the minimal division to player 1, d_1, that guarantees acceptance.

$$d_1 \overbrace{=}^{\geq} \delta_1(1 - d_2)$$

  - Importantly, this is valid at every *even* period, $t = 2, 4, 6, \ldots$ (not only at period $t = 2$).

Bargaining over infinite periods

- Therefore, the division from player 1 to player 2, d_2, and that from player 2 to player 1, d_1, must satisfy

$$d_2 = \delta_2(1 - d_1) \quad \text{and} \quad d_1 = \delta_1(1 - d_2)$$

  - Two equations with two unknowns!

- Plugging one condition inside the other, we have

$$d_2 = \delta_2(1 - \underbrace{(\delta_1(1 - d_2))}_{d_1})$$

- Rearranging,

$$\delta_2 - \delta_2\delta_1 + \delta_2\delta_1 + \delta_2\delta_1 d_2 = d_2$$

and rearranging a little bit more,

$$\delta_2(1 - \delta_1) = d_2(1 - \delta_2\delta_1) \implies d_2^* = \frac{\delta_2(1 - \delta_1)}{1 - \delta_2\delta_1}$$

Bargaining over infinite periods

- And similarly for the division that player 2 makes to player 1,

$$d_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$$

- Hence, in the first period, player 1 makes this offer d_2^* to player 2, who immediately accepts it, since $d_2^* = \delta_2(1-d_1^*)$.
    - (Hence, the game is over after the first stage.)
- Therefore, equilibrium payoffs are:

$$d_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_2\delta_1} \text{ for player 2}$$

Bargaining over infinite periods

- and the equilibrium payoffs for player 1 is:

$$
\begin{aligned}
1 - d_2^* &= 1 - \frac{\delta_2(1-\delta_1)}{1-\delta_2\delta_1} \\
&= \frac{1-\delta_2\delta_1-\delta_2+\delta_2\delta_1}{1-\delta_2\delta_1} \\
&= \frac{1-\delta_2}{1-\delta_2\delta_1}
\end{aligned}
$$

Bargaining over infinite periods

- Note that player 2's payoff, $d_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_2\delta_1}$, *increases* in his own discount factor, δ_2:

$$\frac{\partial\left(\frac{\delta_2(1-\delta_1)}{1-\delta_2\delta_1}\right)}{\partial\delta_2} = \frac{\overbrace{1-\delta_1}^{\geq 0}}{(1-\delta_2\delta_1)^2} \geq 0 \quad \text{since } \delta_1 \in [0,1]$$

- That is, as player 2 assigns more weight to his future payoff, $\delta_2 \rightarrow 1$ (intuitively, he becomes more patient), he gets a larger payoff.
    - That is, as he becomes more patient, he can reject player 1's proposals, and wait until he is the player making proposals.

Bargaining over infinite periods

- In contrast, player 2's payoff, $d_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_2\delta_1}$ *decreases* in the discount factor of player 1 (his opponent), δ_1:

$$\frac{\partial\left(\frac{\delta_2(1-\delta_1)}{1-\delta_2\delta_1}\right)}{\partial\delta_1} = \frac{\delta_2 \overbrace{(\delta_2 - 1)}^{\leq 0}}{(1-\delta_2\delta_1)^2} \leq 0 \text{ since } \delta_2 \in [0,1]$$

- That is, as player 1 assigns more weight to his future payoff, $\delta_1 \to 1$ (intuitively, he becomes more patient), player 2 must offer him a larger share of the pie in order to induce him to accept today.

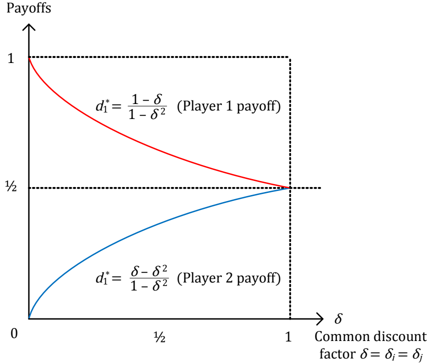
Bargaining over infinite periods

- **Interpretation:** In bargaining games, patience works as a measure of bargaining power:
  - First, if you are more patient, you will not accept low offers from your opponent today, since you can wait until the next period (when you make the offers), and the payoff you get tomorrow (your own offer) is not heavily discounted.
  - Second, a more patient opponent is "more difficult to please" with low offers (since he can simply wait until the next period), and as a consequence, you must make him higher offers in order to achieve acceptance.
- *Bottom line:* the more patient you are (higher δ_i), and the less patient your opponent is (lower δ_j), the larger the share of the pie you keep, and the lower the share he/she keeps in the SPNE of the game.

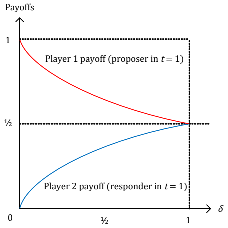
Bargaining over infinite periods

- What if all player are equally patient? (i.e., $\delta_1 = \delta_2 = \delta$)?
  - Then equilibrium payoffs become:

$$d_2^* = \frac{\delta - \delta^2}{1 - \delta^2} \text{ for player 2, and } d_1^* = 1 - d_2^* = \frac{1 - \delta}{1 - \delta^2} \text{ for player 1}$$



Bargaining over infinite periods



- **Interpretation**:
  - When both players are totally impatient ($\delta = 0$), the first player to make an offer gets the entire pie, offering nothing to the responder.
  - When both players are completely patient ($\delta = 1$), they split the surplus evenly.
  - As we move from impatient to patient players, the first player to make an offer reduces his equilibrium payoff, and the responder increases his.