Choice under Uncertainty

Outline

- Lotteries
- Expected Value
- Variance and Standard Deviation
- Expected Utility
- Risk Attitudes
- Measuring Risk
- A Look at Behavioral Economics–Nonexpected Utility

Lotteries

Lotteries

- A lottery is an uncertainty event with N potential outcomes, where each outcome i ocurs with an associated probability $p_i \in [0,1]$, and the sum of these probabilities satisfies $p_1 + p_2 + \cdots + p_N = 1$.
- *Examples*:
  - The act of flipping a coin, with outcomes (heads or tails) each being equally likely.
  - Weather conditions tomorrow, with a different weather outcome associated with a specific probability.
- Probability is the frequency with which we observe a certain outcome of a lottery.

Lotteries

- Lotteries can be understood as probability distributions over outcomes.

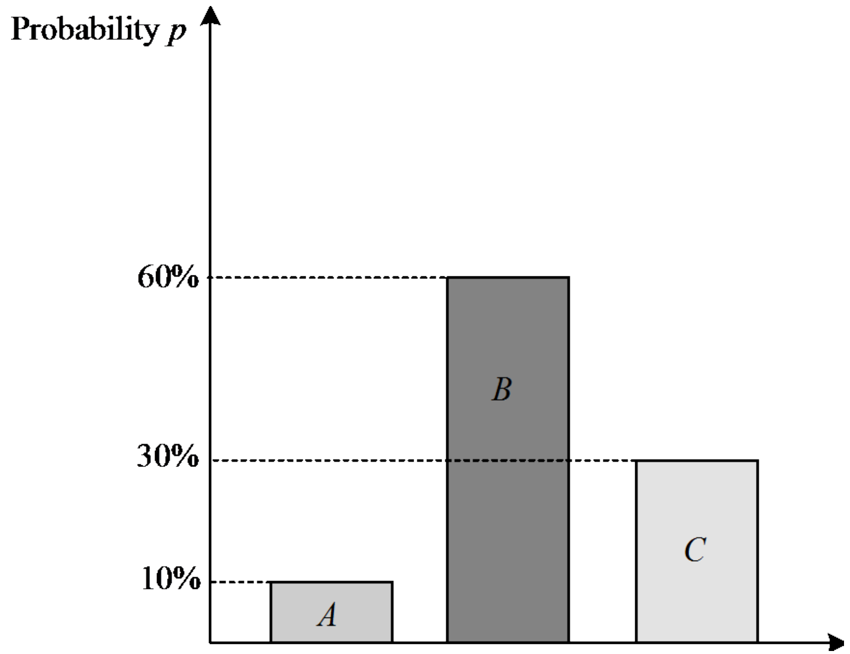

Figure 6.1

Expected Value

Expected Value

- The expected value (EV) is the average payoff of a lottery, where each payoff is weighted by its associated probability.
  - The EV assigns a lager weight to payoffs more likely to occur, and smaller weight to those less likely.
- *Example 6.1: Finding the EV of a lottery.*
  - Consider the probability distribution:
    - Outcome A (\$90) occurs with probability 10%;
    - Outcome B (\$20) occurs with probability 60%;
    - Outcome C (\$60) occurs with probability 30%.
  - The EV of the lottery is given by the weighting average

$$EV = (0.1 \times \$90) + (0.6 \times \$20) + (0.3 \times \$60)$$
$$= 9 + 12 + 18 = \$39.$$

Variance and Standard Deviation

Variance

- The EV informs about the expected payoff of a lottery but it does not measure how risky a lottery is.
- We can find different lotteries with same EV but with different levels of riskiness:

	Outcomes	Probability	Expected Value
Lottery 1 (*Example 6.1*)	A (\$90) B (\$20) C (\$60)	10% 60% 30%	$EV = (0.1 \times \$90) + (0.6 \times \$20) + (0.3 \times \$60) = \$39.$
Lottery 2	a (\$30) b (\$48)	50% 50%	$EV = (0.5 \times \$30) + (0.5 \times \$84) = \$39.$

- One measure of the riskiness of a lottery is its variance.

Variance

- The variance (Var) is the average squared deviation of a lottery from its EV, weighting each squared deviation by the associated probability of that outcome.
- Think about the variance sequentially:
  1. For each possible outcome x, compute $x - EV$:
     - $x - EV > 0$ if payoff x satisfies $x > EV$;
     - $x - EV < 0$ if payoff x satisfies $x < EV$;
     - 0 if payoff x satisfies $x = EV$.
  2. Square this payoff difference, $(x - EV)^2$ so all differences are positive.

Variance

3. Multiply this squared deviation by the probability of the outcome, weighting each outcome with its likelihood of occurring.
4. Repeat steps 1-3 for all possible outcomes, and sum up, obtaining the variance.

- The variance measures the dispersion of the data set relative to its mean.
  - *Example*: A volatile stock has a high variance.
- The variance also increases as outcomes with large squared deviation become more likely.

Variance

- *Example 6.2*: *Finding the variance of a lottery*.
  - The variance of the risky lottery in example 6.1 is

$$Var_{Risky} = 0.1 \times (\$90 - \$39)^2 + 0.6 \times (\$20 - \$39)^2 + 0.3(\$60 - \$29)^2$$
$$= \$609.$$

    - The squared deviation of outcome A, $(\$90 - \$39)^2$ is large, while its probability weight is the lowest (0.1), helping reduce the variance.
    - The squared deviation of outcome B is the smallest, as \$20 is close to the EV of the lottery.
  - The variance of the relative safe lottery is

$$Var_{Safe} = 0.5 \times (\$30 - \$39)^2 + 0.5 \times (\$48 - \$39)^2 = \$81.$$

  - $Var_{Safe} < Var_{Risky}$ because the squared deviations are extremely low.

Standard Deviation

- While the variance helps measure the volatility of data set, it cannot be interpreted as a dollar amount.
- The standard deviation is the square root of the variance,

$$Std = \sqrt{Var}$$

  - It helps understand the dispersion of a data set in dollars, in the original units of payoffs.
- *Example*: For the variances in 6.2,
  - $Std = \sqrt{609} = \$24.67$ for the most risky lottery.
  - $Std = \sqrt{81} = \$9$ for the less risky lottery.
- Std is increasing in Var.

Expected Utility

Expected Utility

- *How to determine which specific lottery a decision maker selects when facing several available lotteries?*
- The expected utility (EU) is the average utility of a lottery, weighting each utility with the associated probability of that outcome.
- EU is similar to EV, as both weight payoffs according to their probabilities by assigning a larger weight to more likely outcomes.
  - EU plugs each payoff into the individual's utility function to better assess how important the payoff is for her.
  - EV considers only payoffs, without evaluating their utility.

Expected Utility

- *Example 6.3: Finding the EU of a lottery.*
  - Consider an individual with utility function $u(I) = \sqrt{I}$, where $I \geq 0$ is the income received in each outcome.
  - The EU of the risky lotter in example 6.1 is

$$EU_{Risky} = \left(0.1 \times \sqrt{\$90}\right) + \left(0.6 \times \sqrt{\$20}\right) + \left(0.3 \times \sqrt{60}\right) = 5.96.$$

  - The EU of the less risky lottery is

$$EU_{Safe} = \left(0.5 \times \sqrt{\$30}\right) + \left(0.5 \times \sqrt{\$48}\right) = 6.20.$$

  - $EU_{Safe} > EU_{Risky}$. While both lotteries generate the same EV, the safer lottery yields a higher EU for this individual.

Risk Attitudes

Risk Aversion

- EU from the less risky lottery:

1. Plot $u(I) = \sqrt{I}$, which is increasing and concave in income (it increases a *decreasing rate*)
2. Place payoff $\$30$ (from outcome A) in the horizontal axis.
3. Extend a vertical line from this point until it hits the utility function at a height of $\sqrt{30} \cong 5.47$ at point A.
4. Repeat steps 2-3 for the other outcome in the lottery with payoff $\$48$.

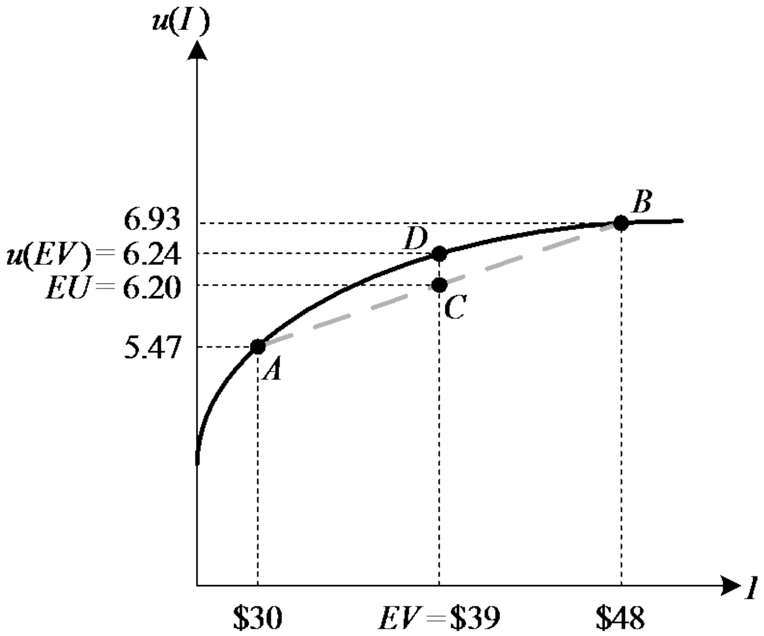

Figure 6.2

Risk Aversion

- EU from the less risky lottery (cont.):

5. Connect points A and B with a line. Because outcomes A and B are equally likely, we find the midpoint (C). The height of this point is the $EU = \$6.20$.

The utility of the EV at point D is

$$u(EV) = \sqrt{EV} = \sqrt{\$39} \cong \$6.24.$$

$u(EV) > EU$, the individual is "risk averse" because she prefers to receive EU with certainty rather than facing the uncertainty of the lottery.

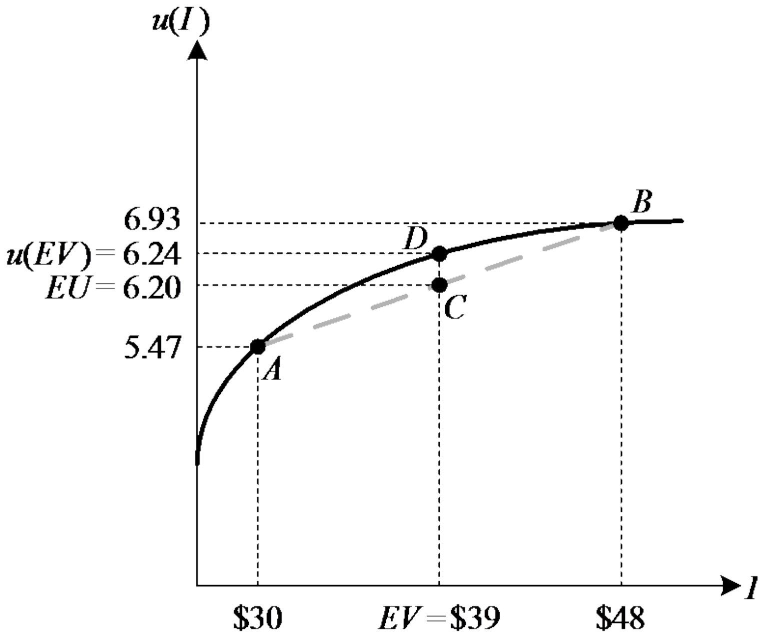

Figure 6.2

Risk Loving

- "Risk lovers" individuals enjoy facing situations with risk.
- *Example 6.4: Finding the EU of a lottery under risk-loving preferences.*
  - Consider individual with $u(I) = I^2$.
  - Find the EU of the two lotteries in example 6.3:

$$EU_{Risky} = (0.1 \times \$90^2) + (0.6 \times \$20^2) + (0.3 \times \$60^2) = 2{,}130.$$

$$EU_{Safe} = (0.5 \times \$30^2) + (0.5 \times \$48^2) = 1{,}602.$$

  - $EU_{Risky} > EU_{Safe}$.

Risk Loving

- EU from the safe lottery:

1. Plot $u(I) = I^2$, which is convex in income (it increases at *increasing rate*)
2. Place payoffs that can arise from the lottery on horizontal axis ($\$30$ and $\$48$).
3. Extend a vertical line upward until we hit the utility function (at height of 900 for point A, and $2{,}304$ for B).
4. Connect points A and B with a line. Find the midpoint C, where $EU = 1{,}602$.

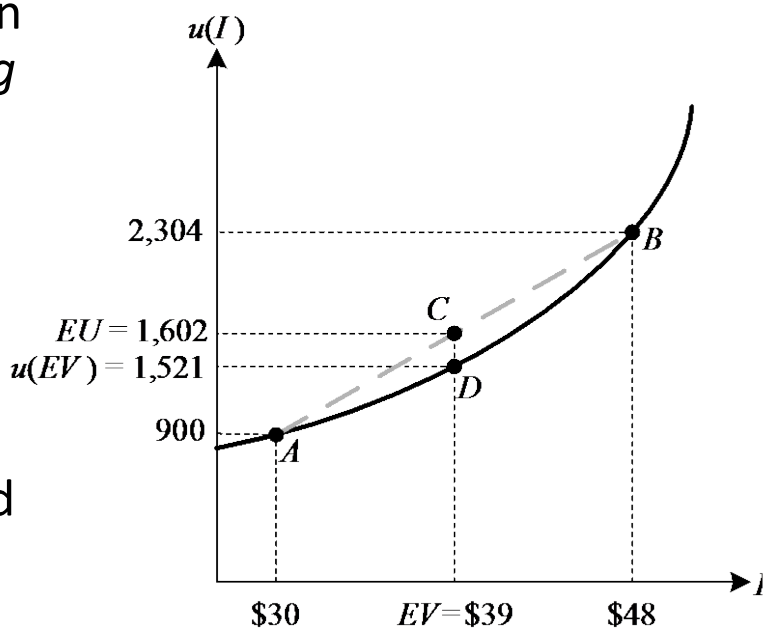

Figure 6.3

Risk Loving

- EU from the safe lottery (cont.):

The utility of EV is found extending a vertical line upward from $EV = \$39$, until we hit the utility function at D.

$$u(\$39) = (39)^2 \cong \$1{,}521.$$

$u(EV) < EU$, the individual is "risk lover" because she prefers to play the lottery and face risk (obtaining EU) to receiving the EV of the lottery with certainty where she obtains $u(EV)$.

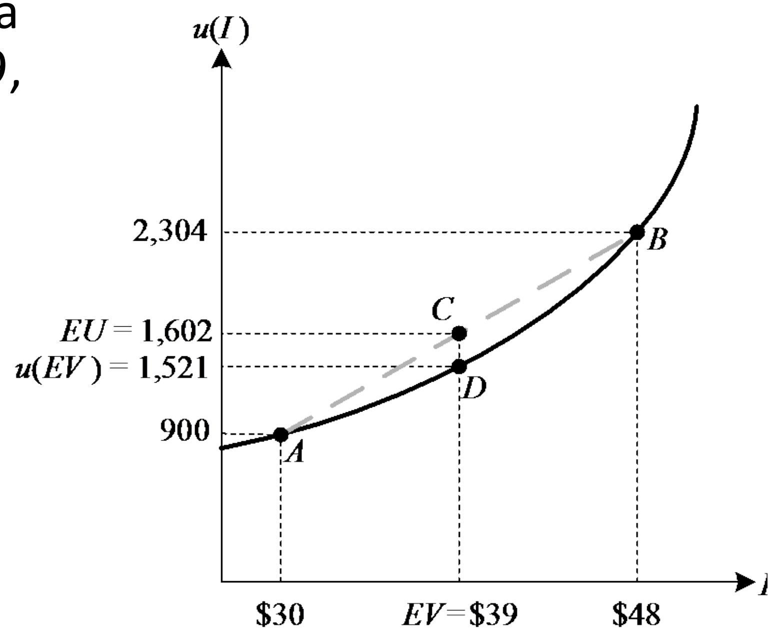

Figure 6.3

Risk Loving

- Convex utility.
  - Risk-loving attitudes emerge when an individual's utility function is convex.
  - Utility functions with the form $u(I) = a + bI^{\gamma}$ are convex if
    - $a, b \geq 0$;
    - $\gamma > 1$.
  - *Examples*:
    - $u(I) = I^2$ where $a = 0$, $b = 1$ and $\gamma = 2$.
    - $u(I) = 5 + 7I^3$.
    - $u(I) = 8 + 2I^5$.

Risk Neutrality

- *Example 6.5: Finding the EU of a lottery under risk-neutral preferences.*
  - Consider individual with $u(I) = I$.
  - The EU from the risky and safe lotteries are

$$EU_{Risky} = (0.1 \times \$90) + (0.6 \times \$20) + (0.3 \times \$60) = 39.$$

$$EU_{Safe} = (0.5 \times \$30) + (0.5 \times \$48) = 39.$$

  - $EU_{Risky} = EU_{Safe}$, she experiences the same EU from the risky and safe lotteries.

Risk Neutrality

- EV and EU from a lottery-risk neutral:

1. Plot $u(I) = I$, which is linear in income (it increases at a constant rate, in this case at 1).
2. Follow the same approach to depict EU of the safe lottery.
3. The height of point C, which is the EU of the lottery, coincides with that of point D, which is the EV of the lottery, $u(EV) = EU$

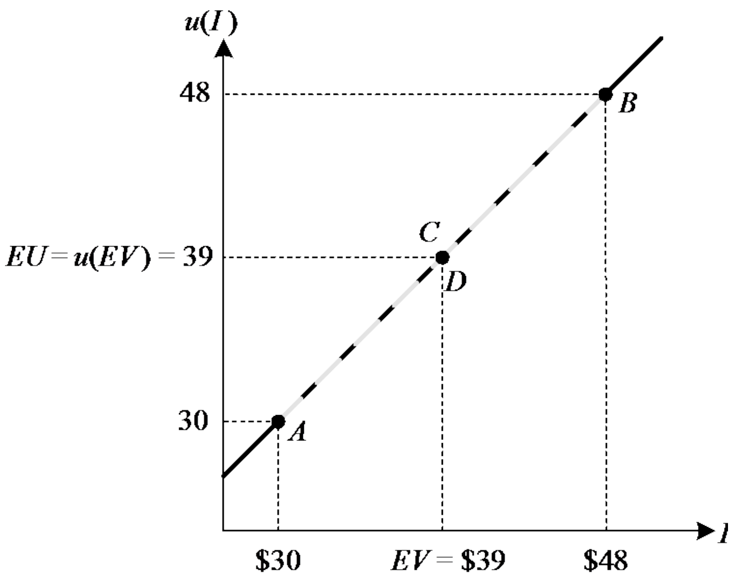

Figure 6.4

The individual is "risk neutral" because she obtains the same utility from receiving the EV of the lottery with certainty, $u(EV)$, and from playing the lottery, EU.

Risk Neutrality

- Linear utility.
  - Risk neutrality arises when an individual's utility function is linear.
  - Linear utility functions take the form $u(I) = a + bI$ were $a, b \geq 0$.
  - *Examples*:
    - $u(I) = I$ where $a = 0$ and b.
    - $u(I) = 3 + 8I$.

Measuring Risk

Risk Premium

- Risk premium (RP) is the amount of money that we need to subtract from the EV in order to make the decision maker indifferent between playing the lottery and accepting the EV from the lottery. RP solves

$$u(EV - RP) = EU.$$

- Consider the scenario of example 6.3 in which you are a risk-averse individual:
  - EU from playing the lottery is $EU = 6.2$.
  - If we offer you the EV of the lottery with certainty is $\$39$, your utility is larger because $u(39) = \sqrt{39} = 6.24$.
  - Knowing that we cut the EV that we offer you by $\$1$, *would you still prefer the* $EV - \$1$ *than the EU? And if we cut the EV in* $\$2$*?*

Risk Premium

- *Example 6.6: Finding the RP of a lottery.*
  - Consider the safe lottery in example 6.3, with $EV = \$39$ and $EU = 6.2$.
  - The RP solves

$$u(39 - RP) = 6.2,$$
$$\sqrt{(39 - RP} = 6.2 \Longrightarrow \left(\sqrt{(39 - RP)}\right)^2 = 6.2^2,$$
$$39 - RP = 6.2^2,$$
$$RP = \$0.56.$$

  - We need to cut the EV of the lottery by $\$0.56$ for the individual to be indifferent between playing the lottery and receiving that (diminished) EV with certainty.
  - If we cut the $EV = \$39$ by more than $\$0.56$ she would prefer playing the lottery rather than the (highly discounted) EV.

Risk Premium

- *Example 6.6* (continued):

  Figure 6.5 illustrates the RP.

  - The diminished EV, after subtracting RP, $EV - RP$, makes the individual indifferent between receiving receiving the amount with certainty and playing the lottery.

  - This diminished EV is also known as the “certainty equivalent.”

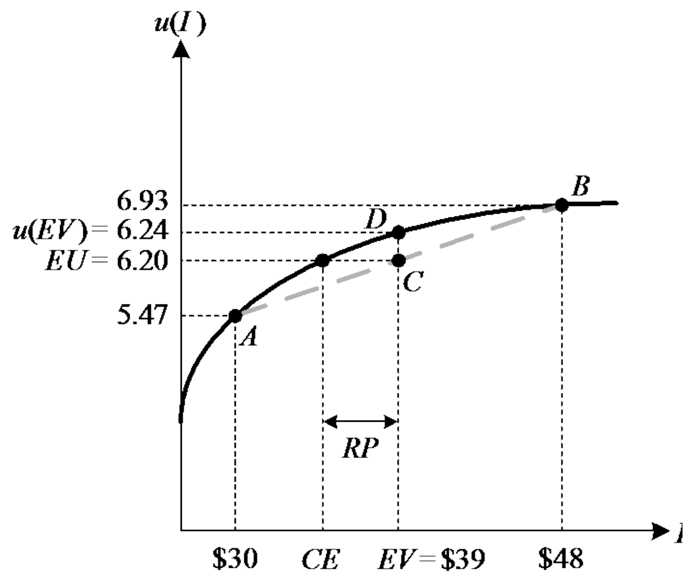

Figure 6.5

Certainty Equivalent

- Risk premium (CE) is the amount of money that, if given to the individual with certainty, makes her indifferent between receiving such a certain amount and playing the lottery,

$$CE = EV - RP.$$

- In example 6.6,

$$CE = EV - RP = 39 - 0.56 = \$38.44.$$

  - If we offer $\$38.44$ to the risk-averse individual, she would be indifferent between receiving this amount and playing the lottery.

Risk Premium

- *Example 6.6: Measuring RP and CE with other risk attitudes.*
  - Consider the risk-living individual in example 6.4.
    - Because $EV = \$39$ and $EU = 1{,}602$. The RP solves

$$u(39 - RP) = 1{,}602,$$
$$(39 - RP)^2 = 1{,}602,$$
$$\sqrt{(39 - RP)^2} = \sqrt{1{,}602},$$
$$39 - RP = 40.2 \Rightarrow RP = -1.02.$$

    - $RP < 0$, for the individual to be indifferent between playing the lottery and receiving a monetary amount with certainty, we would need to offer *more* than the EV.
    - She loves risk, so she would need to be compensated to stop playing the lottery.

Risk Premium

- *Example 6.6* (continued):
  - The CE becomes
    $$CE = EV - RP = 39 - (-1.02) = 40.02.$$
  - $CE > EV$ when the individual is a risk lover.
- Consider now the risk-neutral individual in example 6.5.
  - RP solves
    $$u(39 - RP) = 39,$$
    $$39 - RP = 39,$$
    $$RP = \$0.$$

Risk Premium

- *Example 6.6* (continued):
  - The individual is indifferent between receiving EV with certainty and playing the lottery. So, we don't need to decrease nor increase the EV.
  - The CE becomes

$$CE = EV - RP = EV.$$

Summary of risk aversion measures

Table 6.1

	Risk Averse	Risk Lover	Risk Neutral
Utility function	Concave	Convex	Linear
$U(EV)$ vs. EU	$u(EV) > EU$	$u(EV) < EU$	$u(EV) = EU$
Risk Premium, RP	$+$	$-$	0
Certainty Equivalent, CE	$CE < EV$	$CE > EV$	$CE = EV$
Arrow-Pratt coefficient, AP	$AP > 0$	$AP < 0$	$AP = 0$
Exponent γ in $u(I) = a + bI^{\gamma}$	Between 0 and 1	Larger than 1	1

A Look at Behavioral Economics–Nonexpected Utility

Nonexpected Utility

- The EU measure is tractable and intuitive, and can be "experimentally tested."
  - In experiments individuals are asked to sit at computer terminals and choose among lotteries.
  - Monetary incentives are provided.
- Experiments have found that participants sometimes behaved differently from what EU would have predicted.
- In the field of behavioral economics, researchers have proposed alternative theories of decision-making under under uncertainty to account for these experimental anomalies.

Nonexpected Utility

- *Example 6.9: The certainty effect.*
  - Kahneman and Tversky (1979) asked experimental participants to consider what decision they would make in 2 choices:
    - *Choice 1*:
      a) Lottery A: Receive $\$3{,}000$ with certainty.
      b) Lottery B: Receive \$4,000 with probability 0.8 and $\$0$ with probability 0.2.
    - *Choice 2:*
      c) Lottery C: Receive $\$3{,}000$ with probability 0.25 and $\$0$ with probability 0.75.
      d) Lottery D: Receive $\$4{,}000$ with probability 0.20 and receive $\$0$ with probability 0.8.
  - Most participants preferred lottery A over B in Choice 1 and lottery D over C in Choice 2.

Nonexpected Utility

- *Example 6.9* (continued):
  - However, these preferences are inconsistent with EU theory.
  - An individual prefers lotter A over B in *Choice 1* if and only if

$$u(\$3{,}000) > 0.8u(4{,}00) + 0.2u(\$0).$$

  - An individual prefers lottery D over C in *Choice 2* if and only if

$$0.2u(\$4{,}000) + 0.8u(\$0) > 0.25u(\$3{,}000) + 0.75u(\$0),$$

$$[0.2u(\$4{,}000) + 0.8u(\$0)]/0.25 > [0.25u(\$3{,}000) + 0.75u(\$0)]/0.25,$$

$$0.8u(\$4{,}000) + 0.2u(\$0) > u(3{,}000).$$

Nonexpected Utility

- *Example 6.9* (continued):
  - The inequality in *Choice 2*

    $$0.8u(\$4{,}000) + 0.2u(\$0) > u(3{,}000).$$

    is the opposite of the inequality in *Choice 1,*

    $$u(\$3{,}000) > 0.8u(4{,}00) + 0.2u(\$0).$$

  - This result is problematic because we did not assume any risk attitude.
  - We cannot rationalize these choices using EU, regardless of the utility function of the individual.

Weighted Utility

- An individual with weighted utility (WU) assigns to each payoff x in the lottery, a weight $g(x)$, where $g(x) \neq g(y)$.
- Consider a lottery between payoff x with probability p and payoff y with probability $1 - p$.
- The EU of this lottery is

$$EU = pu(x) + (1 - p)u(y),$$

  where $u(x)$ and $u(y)$ are the utilities from payoffs x and y. And p and $(1 - p)$ are the probability weight on payoff x and y.
- WU only changes probabilities weight as follows

$$WU = \underbrace{\frac{g(x)}{g(x)p + g(y)(1 - p)}}_{\text{Prob. weight on payoff } x} u(x) + \underbrace{\frac{g(y)(1 - p)}{g(x)p + g(y)(1 - p)}}_{\text{Prob. weight on payoff } y} u(y)$$

Weighted Utility

- When $g(x) = g(y)$,

$$WU = pu(x) + (1 - p)u(y)$$
$$= EU.$$

- When $g(x) \neq g(y)$, WU theory and EU theory yield different results. When $g(y) > g(x)$, if $y > x$, the WSU assigns larger importance to the upward outcome, yielding $WU > EU$.
  - The individual is more willing to participate in the lottery when she evaluates it according to the lottery's WU than its EU.

Weighted Utility

- *Example 6.10:* ***Weighted utility.***
  - Consider the safe lottery in example 6.3, with payoffs $x = \$30$ and $y = \$48$, both occurring with probability $1/2$.
  - The individual's utility function was $u(x) = \sqrt{x}$. The safe lottery generated $EU = 6.20$.
  - If this lottery is evaluated according to WU and $g(x) = 2$, while $g(y) = 3$, her WU becomes

$$WU = \frac{2\frac{1}{2}}{2\frac{1}{2} + 3\frac{1}{2}}\sqrt{\$30} + \frac{3\frac{1}{2}}{2\frac{1}{2} + 3\frac{1}{2}}\sqrt{\$48}$$

$$= \left(\frac{2}{5} \times 5.47\right) + \left(\frac{3}{5} \times 6.92\right) = 6.34 > EU.$$

    - She assigns a larger weight to the upward of the lottery (payoff y), finding it more attractive when evaluating it according to WU.

Weighted Utility

- *Example 6.11: Using WU to explain the certainty effect.*
  - Consider the individual in example 6.10.
  - We check whether her preferences can explain the certainty effect in example 6.9.
  - Lottery A is preferred to B in *Choice 1* if and only if

$$\sqrt{\$3{,}000} > \frac{2 \times 0.2}{(3 \times 0.8) + (2 \times 0.2)}\sqrt{\$0} + \frac{3 \times 0.8}{(3 \times 0.8) + (2 \times 0.2)}\sqrt{\$4{,}000},$$

$$54.77 > 54.21.$$

Weighted Utility

- *Example 6.11* (continued):
  - Lottery D is preferred to C in *Choice 2* if and only if

$$\frac{2 \times 0.2}{(3 \times 0.8) + (2 \times 0.2)} \sqrt{\$0} + \frac{3 \times 0.8}{(3 \times 0.8) + (2 \times 0.2)} \sqrt{\$4{,}000}$$

$$> \frac{3 \times 0.75}{(3 \times 0.75) + (2 \times 0.25)} \sqrt{\$3{,}000} + \frac{2 \times 0.75}{(3 \times 0.75) + (2 \times 0.25)} \sqrt{\$0},$$

$$54.21 > 44.81.$$

  - Therefore, the experimental observations in Kahneman and Tversky (1979) can be explained by WU theory.

Prospect Theory

- Tversky and Kahneman (1986) proposed that the value than an individual obtain from a lottery can be different from the EU.
- Consider the lottery with two payoff, x with probability p, and y with probability $1-p$.
- The value of the lottery is

$$V = w(p)v(x, x_0) + w(1-p)v(y, x_0)$$

- This value of the lottery differs from the EU in three dimensions:
    1. Probability weights.
    2. The use of reference points.
    3. Loss aversion.

Prospect Theory

1. *Probability weights*.
   - Probability is weighted with the probability weighting function $w(p)$, rather than considering p directly.
     - When $w(p) > p$, the individual overestimates the likelihood of outcome x.
     - When $w(p) < p$, the individual underestimates the likelihood of outcome x.
     - When $w(p) = p$, the individual assigns the same probability weights as when she uses EU to evaluate a lottery.
   - A similar argument applies to outcome y, with weighted probability $w(1-p)$.

Prospect Theory

2. *The use of reference points*.
   - Every payoff x is evaluated against a reference point x_0 (status quo).
   - Individual's utility from payoff x is $v(x, x_0)$, and from payoff y is $v(y, x_0)$.
   - Utility $v(x, x_0)$ is increasing in x, and concave in all payoffs that lie above the reference point, $x > x_0$.
     - The individual is risk averse toward gains (relative to the reference point).
   - Utility $v(x, x_0)$ is convex for all payoffs that lie below the reference point, $x < x_0$.
     - The individual is risk loving toward losses.

Prospect Theory

2. *The use of reference points* (cont.).

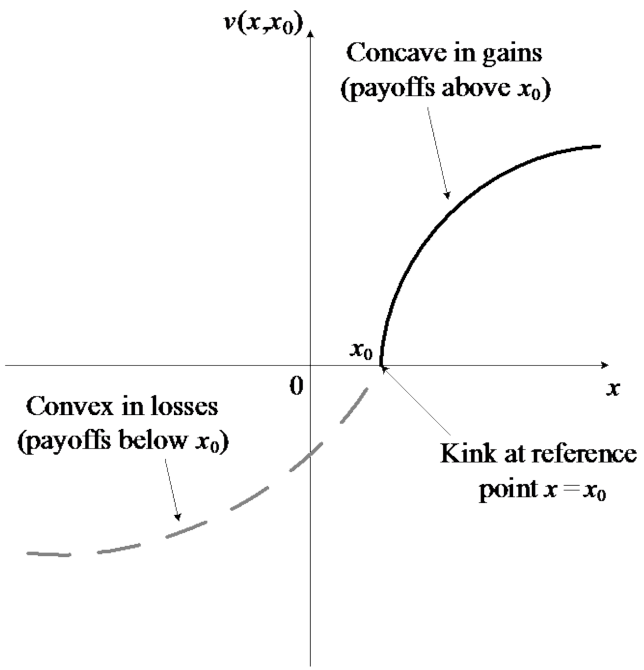

Figure 6.6

Prospect Theory

3. *Loss aversion*.
   - Utility $v(x, x_0)$ has a kink at reference point x_0, rather than a smooth transition.
     - She suffers a large disutility when she losses $\$1$ relative to the reference point x_0, which is referred as the individual exhibiting loss aversion.

Prospect Theory

- *Example 6.12: Prospect Theory.*
  - Consider probability weighting function $w(p) = \frac{p^{1/2}}{p^{1/2}+(1-p)^{1/2}}$.

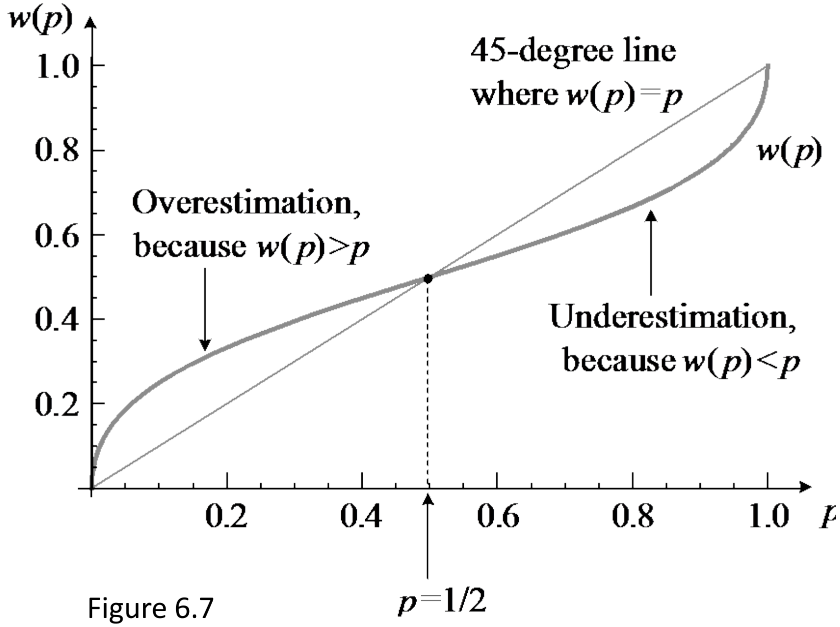

Figure 6.7

Prospect Theory

- *Example 6.12* (continued):
  - Regarding utility function $v(x, x_0)$, consider reference point $x_0 = \$0$,

$$v(x, 0)\begin{cases} x^{1/2} \text{ for all } x \geq 0, \\ -3(-x)^{1/2} \text{ for all x} < 0. \end{cases}$$

  - The kink happens at the origin, $x = 0$.
  - The individual has:
    - Concave utility function $x^{1/2}$ for all positive payoffs.
      - Exponent $1/2$ captures her concavity in gains.
    - Convex utility function $-3(-x)^{1/2}$ for all negative payoffs.
      - Exponent $1/2$ capture her convexity in gains;
      - -3 represents her loss aversion.
      - If the utility for losses was $-x^{1/2}$, gain and losses would produce the same effect on utility, leading to no kink.

Prospect Theory

- *Example 6.13: Using prospect theory to explain the certainty effect.*
  - Consider preferences por lotteries in example 6.9 (A over B in *Choice 1*, and D over C in *Choice 2*),
  - And probability weighting function $w(p) = \frac{p^{1/2}}{p^{1/2}+(1-p)^{1/2}}$ and utility function with reference point $x_0 = 0$, from example 6.12.
  - Kahneman and Tyversky (1979) found that individuals prefer lottery A over B in *Choice 1*,

$$\$3{,}00^{1/2} > \frac{0.8^{1/2}}{0.8^{1/2} + (1-0.8)^{1/2}} \$4{,}000^{1/2} + \left(1 - \frac{0.8^{1/2}}{0.8^{1/2} + (1-0.8)^{1/2}}\right) \$0^{1/2},$$

$$\$54.77 > \$42.16.$$

Prospect Theory

- *Example 6.13* (continued):
  - Individuals prefer lottery D over C in *Choice 2*,

$$\frac{0.2^{1/2}}{0.2^{1/2}+(1-0.2)^{1/2}}\$4{,}000^{1/2}+\left(1-\frac{0.2^{1/2}}{0.2^{1/2}+(1-0.2)^{1/2}}\right)\$0^{1/2}$$
$$>\frac{0.25^{1/2}}{0.25^{1/2}+(1-0.25)^{1/2}}\$4{,}000^{1/2}+\left(1-\frac{0.25^{1/2}}{0.25^{1/2}+(1-0.25)^{1/2}}\right)\$0^{1/2},$$
$$\$21.08>\$20.05.$$

  - Lottery preferences in *Choice 1* and *Choice 2* are consistent with prospect theory.