EconS 424 - Homework #5 (Due on Friday April 23rd)

Question #1

Consider two firms competing a la Cournot, and facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. For simplicity, assume that firms face a common marginal cost of production $c = 10$.

a) *Unrepeated game.* Find the equilibrium output each firm produces when competing a la Cournot (that is, when they simultaneously and independently choose their output levels) in the unrepeated version of the game (that is, when firms interact only once). In addition, find the profits that each firm earns in equilibrium.

b) *Repeated game - Collusion.* Assume now that the CEOs from both companies meet to discuss a collusive agreement that would increase their profits. Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the output level that each firm should select to maximize joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

c) *Repeated game – Permanent punishment.* Consider a grim-trigger strategy in which every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. Otherwise, every firm deviates to the Cournot equilibrium thereafter (that is, every firm produces the Nash equilibrium of the unrepeated game found in part a forever). In words, this says that the punishment of deviating from the collusive agreement is *permanent*, since firms never return to the collusive outcome. For which discount factors this grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

d) *Repeated game – Temporary punishment.* Consider now a "modified" grim-trigger strategy. Like in the grim-trigger strategy of part (c), every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. However, if a deviation is detected by either firm, every firm deviates to the Cournot equilibrium during only 1 period, and then every firm returns to cooperation (producing the collusive output). Intuitively, this implies that the punishment of deviating from the collusive agreement is now *temporary* (rather than permanent) since it lasts only one period. For which discount factors this "modified" grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

e) Consider again the temporary punishment in part (d), but assume now that it lasts for two periods. How are your results from part (d) affected? Interpret.

f) Consider again the temporary punishment in part (d), but assume now that it lasts for three periods. How are your results from part (d) affected? Interpret.

Answer:

Part (a). Firm i's profit function is given by:

$$\pi_i(q_i, q_j) = \left[100 - \left(q_i + q_j\right)\right]q_i - 10q_i \qquad (7)$$

Differentiating with respect to output q_i yields,

$$100 - 2q_i - q_i - 10 = 0$$

Solving for q_i, we find firm *i*'s best response function

$$q_i(q_j) = 45 - \frac{q_j}{2} \qquad \text{(BRF}_7\text{)}$$

Since this is a symmetric game, firm *j*'s best response function is symmetric. Therefore, in a symmetric equilibrium both firms produce the same output level, $q_i = q_j$, which helps us rewrite the above best response function as follows

$$q_i = 45 - \frac{q_i}{2}$$

Solving this expression yields equilibrium output of $q_i^* = q_1^* = q_2^* = 30$. Substituting these results into profits $\pi_1(q_1, q_2)$ and $\pi_2(q_1, q_2)$, we obtain that

$$\pi_i(q_i^*, q_j^*) = \pi_i(30,30) = [100 - (30 + 30)]30 - 10 \times 30 = \$900$$

where firm's equilibrium profits coincide, that is, $\pi_i(q_i^*, q_j^*) = \pi_1(q_1^*, q_2^*) = \pi_2(q_1^*, q_2^*)$.

In summary, equilibrium profit is \$900 for each firm and combined profits are \$900+\$900=\$1800.

Part (b). In this case, firms maximize the sum $\pi_1(q_1, q_2) + \pi_2(q_1, q_2)$, as follows

$$\{[100 - (q_i + q_j)]q_i - 10q_i\} + \{[100 - (q_i + q_j)]q_j - 10q_j\} \qquad (8)$$

Differentiating with respect to output q_i, yields

$$100 - 2q_i - q_j - 10 - q_j = 0$$

Solving for q_i, we obtain

$$q_i(q_j) = 45 - q_j$$

and similarly when we differentiate with respect to q_j. In a symmetric equilibrium both firms produce the same output level, $q_i = q_j$, which helps us rewrite the above expression as follows

$$q_i = 45 - q_i$$

Solving for output q_i we obtain equilibrium output $q_i^* = q_1^* = q_2^* = \frac{45}{2} = 22.5$ units. This yields each firm a profit of

$$\pi_i(q_i^*, q_j^*) = \pi_i(22.5,22.5)$$

$$= \{[100 - (22.5 + 22.5)]22.5 - (10 \times 22.5)\} + \{[100 - (22.5 + 22.5)]22.5 - (10 \times 22.5)\}$$

$$= \$1012.5$$

Thus, each firm makes a profit of $1,012.5, which yields a joint profit of $2,025.

Part (c). For this part of the exercise, let us first list the payoffs that the firm can obtain from each of its output decisions:

- Cooperation yields a payoff of $1,012.5 for firm i.
- Defecting while firm j cooperates $\left(q_j = 22.5\right)$, by choosing the Cournot output $(q_i = 30)$ yields a profit of $1,125 for firm *i*. However, such defection is punished with Cournot competition in all subsequent periods, which yields a profit of only $900.

Therefore, firm *i* cooperates as long as

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + \cdots \geq 1125 + 900\delta + 900\delta^2 + \cdots$$

which can be simplified to

$$1012.5(1 + \delta + \delta^2 + \cdots) \geq 1125 + 900\delta(1 + \delta + \delta^2 + \cdots)$$

and expressed more compactly as

$$\frac{1012.5}{1-\delta} \geq 1125 + \frac{900\delta}{1-\delta}$$

Multiplying both sides of the inequality by $1 - \delta$, yields

$$1012.5 \geq 1125(1-\delta) + 900$$

which rearranging and solving for discount factor δ entails

$$\delta \geq 0.5$$

Thus, for cooperation to be sustainable in an infinitely repeated game with permanent punishments, firms' discount factor δ has to be at least 0.5. In words, firms must put a sufficient weight on future payoffs.

Part (d). The set up is analogous to that in part (c) of the exercise, but we now write that cooperation is possible if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 \ldots . \geq 1125 + 900\delta + 1012.5\delta^2 \ldots .$$

Importantly, note that payoffs after the punishment period (in this case, a one period punishment of Cournot competition with $900 profits) returns to cooperation, explaining that all payoffs in the third term of the left-hand and right-hand side of the inequality coincide. We can therefore cancel them out, simplifying the above inequality to

$$1012.5 + 1012.5\delta \geq 1125 + 900\delta$$

Rearranging yields

$$112.5\delta \geq 112.5$$

$$\Rightarrow \delta \geq 1$$

ultimately simplifying to $\delta \geq 1$. That is, when deviations are only punished during one period cooperation can be sustained if firms assign the same value to current than future payoffs, which does not generally occur.

Part (e). Following a similar approach as in part (d), we write that cooperation can be sustained if and only if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 \ldots . \geq 1125 + 900\delta + +900\delta^2 + 1012.5\delta^3 \ldots.$$

The stream of payoffs after the punishment period (in this case, a two-period punishment of Cournot competition with $900 profits) returns to cooperation. Hence, the payoffs in both the left- and right-hand side of the inequality after the punishment period coincide and cancel out. The above inequality then becomes

$$1012.5 + 1012.5\delta + 1012.5\delta^2 \geq 1125 + 900\delta + 900\delta^2$$

which, after rearranging, yields

$$\delta^2 + \delta - 1 \geq 0$$

ultimately simplifying to $\delta \geq 0.62$. In words, when deviations are punished for two periods, cooperation can be sustained if firms' discount factor δ is at least 0.62.

Part (f). Following a similar approach as in part (e), we write that cooperation can be sustained if and only if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 + 1012.5\delta^4 \ldots . \geq 1125 + 900\delta + 900\delta^2 + 900\delta^3 + 1012.5\delta^4 \ldots.$$

The stream of payoffs after the punishment period (in this case, a three-period punishment of Cournot competition with $900 profits) returns to cooperation. Hence, the payoffs in both the left- and right-hand side of the inequality after the punishment period coincide and cancel out. The above inequality then becomes

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 \geq 1125 + 900\delta + 900\delta^2 + 900\delta^3$$

which, after rearranging, yields

$$\delta^3 + \delta^2 + \delta - 1 \geq 0$$

ultimately simplifying to $\delta \geq 0.54$. Therefore, when deviations are punished for two periods, cooperation can be sustained if firms' discount factor δ is at least 0.54.

<u>Question #2</u>

Here is a description of the simplest poker game. There are two players and only two cards in the deck, an Ace and a King. First, the deck is shuffled and one of the two cards in dealt to player 1. That is, nature chooses the card for player 1: being the Ace with probability 2/3 and the King with probability 1/3. Player 2 has previously received a card, which both players had a chance to observe. Hence, the only uninformed player is player 2, who does not know whether his opponent has received a King or an Ace. Player 1 observes his card and then chooses whether to bet (B) or fold (F). If he folds, the game ends, with player 1 obtaining a payoff of −1 and player 2 getting a payoff of 1 (that is, player 1 loses his ante to player 2). If player 1 bets, then player 2 must decide whether to respond betting or folding. When player 2 makes this decision, she knows that player 1 bets, but she has not observed player 1's card. The game ends after player 2's action. If player 2 folds, then the payoff vector is (1, −1), meaning player 1 gets 1 and player 2 gets −1, regardless of player 1's hand. If player 2, instead, responds betting, then the payoff depends on player 1's card: if player 1 holds the Ace then the payoff vector is (2, −2), thus indicating that player 1 wins; if player 1 holds the King, then the payoff vector is (−2, 2), reflecting that player 1 looses. Represent this game in the extensive form and in the Bayesian normal form, and find the Bayesian Nash Equilibrium (BNE) of the game.

Answer

Figure 1 depicts the game tree of this incomplete information game.

Player 2 has only two available strategies $S_2 = \{Bet, Fold\}$, but player 1 has four available strategies $S_1 = \{Bb, Bf, Fb, Ff\}$. In particular, for each of his strategies, the first component represents player 1's action when the card he receives is the Ace while the second component indicates his action after receiving the King. This implies that the Bayesian normal form representation of the game is given by the following 4 × 2 matrix (Table 1).

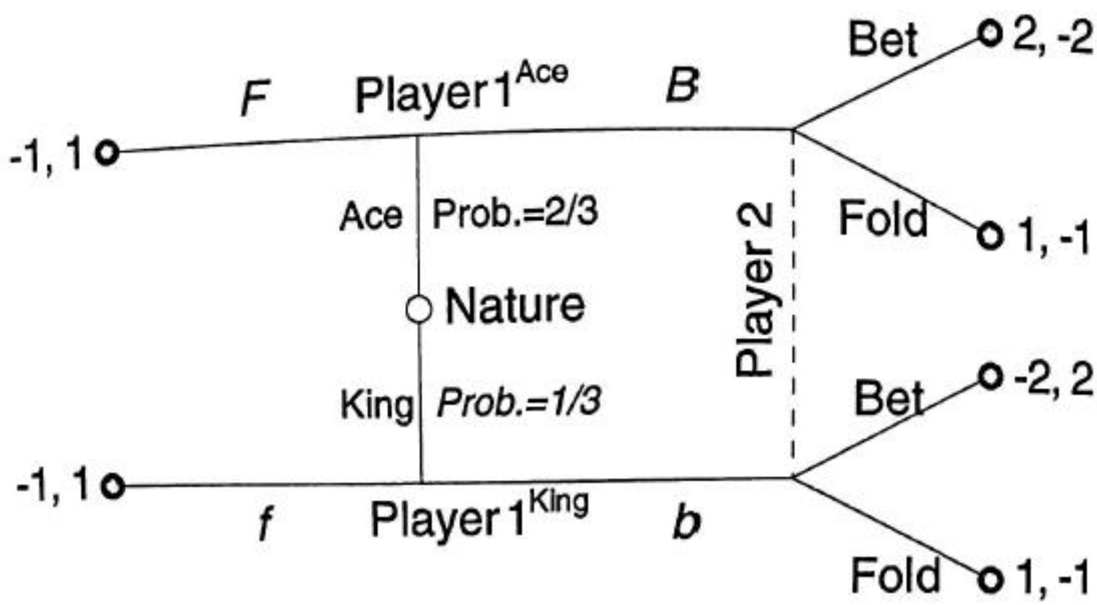

Fig. 1 A simple poker game

		Player 2	
		Bet	Fold
Player 1	Bb		
	Bf		
	Fb		
	Ff		

Table 1 Strategies in the Bayesian normal form representation of the poker game

In order to find the expected payoffs for strategy profile (*Bb*, *Bet*), i.e., in the top left-hand side cell of the matrix, we proceed as follows:

$$EU_1 = \frac{2}{3}*2 + \frac{1}{3}*(-2) = \frac{2}{3} \text{ and } EU_2 = \frac{2}{3}*(-2) + \frac{1}{3}*(2) = -\frac{2}{3}$$

Similarly for strategy profile (*Bf*, *Bet*),

$$EU_1 = \frac{2}{3}*(-1) + \frac{1}{3}*(2) = 0 \text{ and } EU_2 = \frac{2}{3}*(-2) + \frac{1}{3}*1 = -1$$

For strategy profile (*Fb*, *Bet*),

$$EU_1 = \frac{2}{3}(-1) + \frac{1}{3}(-2) = -\frac{4}{3} \text{ and } EU_2 = \frac{2}{3}*1 + \frac{1}{3}*2 = \frac{4}{3}$$

For strategy profile (*Ff*, *Bet*), in the bottom left-hand side cell of the matrix, we obtain

$$EU_1 = \frac{2}{3}(-1) + \frac{1}{3}(-1) = -1 \text{ and } EU_2 = \frac{2}{3}*1 + \frac{1}{3}*1 = 1$$

In strategy profile (*Bb*, *Fold*), located in the top right-hand side cell of the matrix, we have

$$EU_1 = \frac{2}{3}*1 + \frac{1}{3}*1 = 1 \text{ and } EU_2 = \frac{2}{3}(-1) + \frac{1}{3}(-1) = -1$$

For strategy profile (*Bf*, *Fold*),

$$EU_1 = \frac{2}{3}*1 + \frac{1}{3}(-1) = \frac{1}{3} \text{ and } EU_2 = \frac{2}{3}(-1) + \frac{1}{3}*1 = -\frac{1}{3}$$

Similarly for (*Fb*, *Fold*),

$$EU_1 = \frac{2}{3}*(-1) + \frac{1}{3}*1 = -\frac{1}{3} \text{ and } EU_2 = \frac{2}{3}*1 + \frac{1}{3}(-1) = \frac{1}{3}$$

Finally, for (*Ff*, *Fold*), in the bottom right-hand side cell of the matrix

$$EU_1 = \frac{2}{3}*(-1) + \frac{1}{3}(-1) = -1 \text{ and } EU_2 = \frac{2}{3}*1 + \frac{1}{3}*1 = 1$$

We can now insert these expected payoffs into the Bayesian normal form representation (Table

2).

		Player 2	
		Bet	Fold
Player 1	Bb	2/3, −2/3	1, −1
	Bf	1, −1	1/3, −1/3
	Fb	−4/3, 4/3	−1/3, 1/3
	Ff	−1, 1	−1, 1

Table 2 Bayesian form representation of the poker game after inserting expected payoffs

We can now identify the best response to each player.

- For player 2, his best response when player 2 bets (in the left-hand column) is to play *Bf* since it yields a higher payoff, i.e., 1, than any other strategy, i.e., $BR_1(Bet) = Bf$. Similarly, when player 2 chooses to fold (in the right-hand column), player 1's best response is to play *Bb*, given that its associated payoff, $1, exceeds that of all other strategies, i.e., $BR_1(Fold) = Bb$. Hence, the best responses yield an expected payoff of $\frac{2}{3}$ and 1, respectively; as depicted in the underlined payoffs of Table 3.
- Similarly, for player 2, when player 1 chooses *Bb* (in the top row), his best response is to bet, since his payoff, $\frac{-2}{3}$, is larger than that from folding, −1, i.e., $BR_2(Bb) = Bet$. If, instead, player 1 chooses *Bf* (in the second row), player 2's best response is to fold, $BR_2(Bf) = Fold$, since his payoff from folding, −1/3, is larger than from betting, −1. A similar argument applies to the case in which player 1 chooses *Fb*, where player 2 responds betting, $BR_2(Fb) = Bet$, obtaining a payoff of $\frac{4}{3}$. Finally, when player 1 chooses *Ff* in the bottom row, player 2 is indifferent between responding with bet or fold since they both yield the same payoff ($1), i.e., $BR_2(Ff) = \{Bet, Fold\}$.

Hence, there is no BNE in pure strategies.