EconS 424- Strategy and Game Theory
Homework #4 - Due date: Wednesday, April 7th.

1. **Exercises from Harrington's textbook:**

   (a) Chapter 9: exercises 6, 8, and 9.

Solution Exercise 6

a. Derive the strategy set for each player. (*Note:* If you do not want to list all of the strategies, you can provide a general description of a player's strategy, give an example, and state how many strategies are in the strategy set.)

ANSWER: A strategy for a player specifies an action for each information set. A strategy for player 1 assigns an action to the initial node, an action when player 1 chose c_1 and player 2 chose a_2, and an action when player 1 chose c_1 and player 2 chose b_2. We will then represent a strategy for player 1 as a triple of actions in which the ordering corresponds to the sequence of information sets just given. For player 2, a strategy assigns an action when player 1 chose a_1, an action when player 1 chose b_1, and an action when player 1 chose c_1. As player 3 has only one information set, a strategy for player 3 is a single action. Player 1 has $3 \times 2 \times 2 = 12$ strategies, player 2 has $2 \times 2 \times 2 = 8$ strategies, and player 3 has 2 strategies.

b. Derive all subgame perfect Nash equilibria.

ANSWER: (1) Let's derive Nash equilibria of the subgame associated with player 1 having chosen a_1. The strategic form is shown in the figure below. It has two Nash equilibria: $(a_2,b_3)$and (b_2,a_3).

Player 2 \ Player 3	a_3	b_3
a_2	0,0	2,2
b_2	1,2	1,1

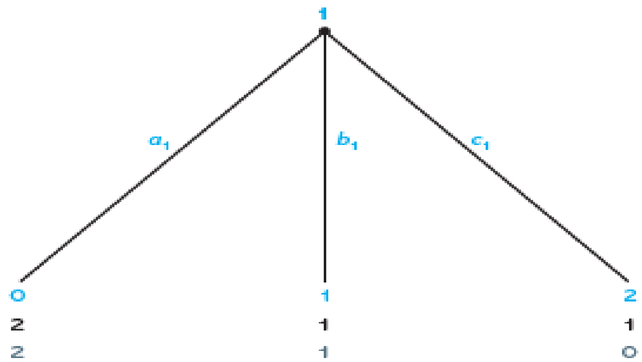
(2) In the subgame associated with player 1 having chosen b_1, player 2 will choose a_2; that is the Nash equilibrium.

(3) In the subgame associated with player 1 having chosen c_1 and player 2, a_2, player 1 chooses d_1.

(4) In the subgame associated with player 1 having chosen c_1 and player 2, b_2, player 1 chooses d_1.

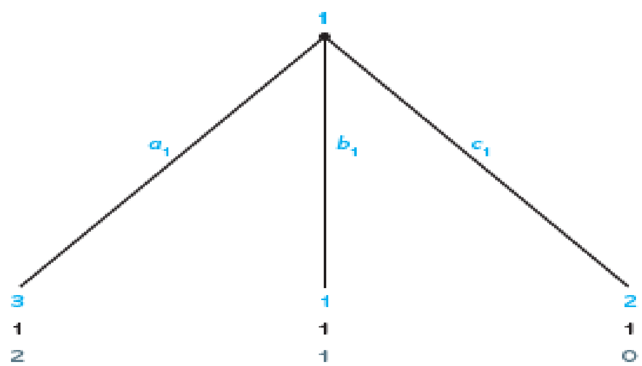
(5) Inserting the equilibrium payoffs for (3)–(4) in the subgame associated with player 1 having chosen c_1, player 2 chooses a_2.

Using (1)–(5), we can solve for the final subgame, which is the game itself. If (a_2, b_3) is the Nash equilibrium of the game associated with player 1 having chosen a_1, then the extensive form is as shown in the figure below:



Then player 1 chooses c_1. Hence, we have constructed a subgame perfect Nash equilibrium: $(c_1/d_1/d_1, a_2/a_2/a_2, b_3)$.

Now suppose (b_2, a_3) is the Nash equilibrium of the game associated with player 1 having chosen a_1. The extensive form is shown in the figure below:



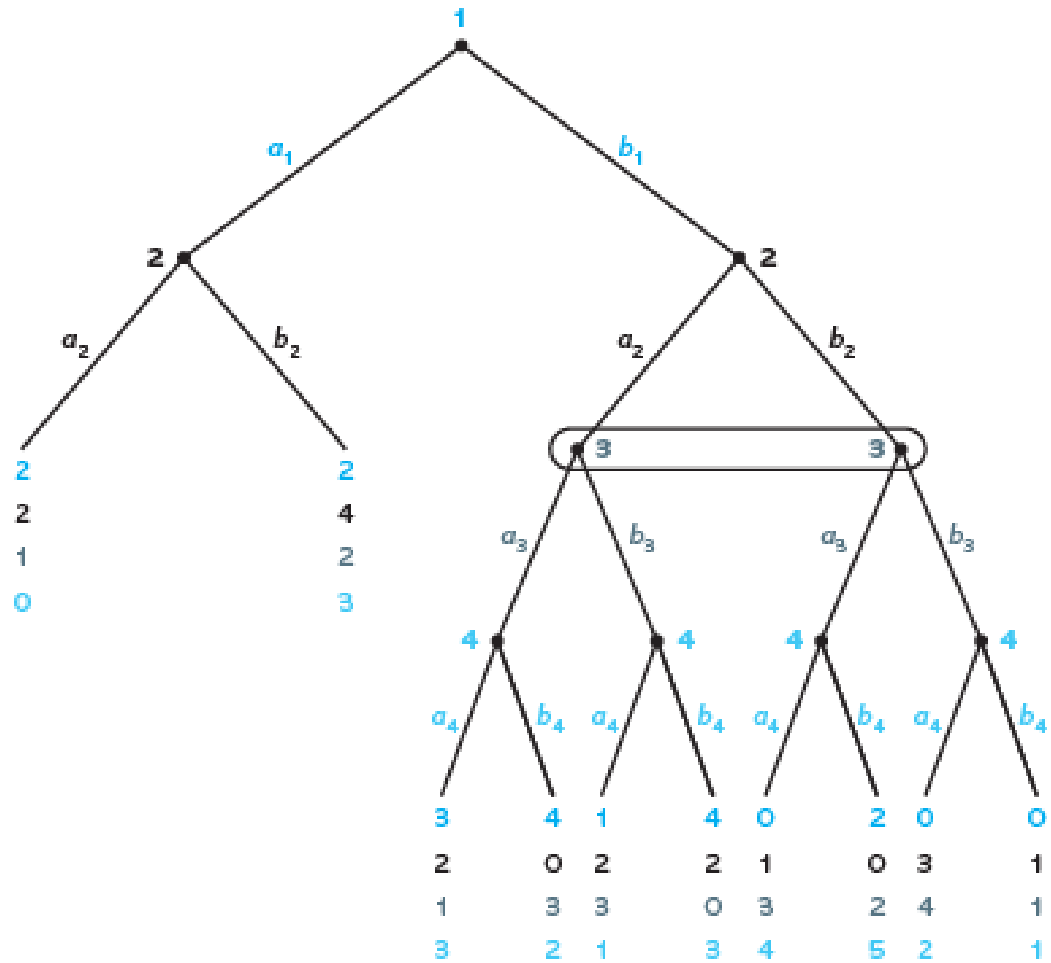
Player 1 chooses a_1 at the initial node. Hence, a second subgame perfect Nash equilibrium is $(a_1/d_1/d_1, b_2/a_2/a_2, a_3)$.

c. Derive a Nash equilibrium that is not an SPNE, and explain why it is not an SPNE.

ANSWER: We know $(a_1/d_1/d_1, b_2/a_2/a_2, a_3)$ is a subgame perfect Nash equilibrium. Let us change only one action of player 1 so that the strategy profile is $(a_1/d_1/e_1, b_2/a_2/a_2, a_3)$. This is a Nash equilibrium (it is enough to check that this derives a Nash equilibrium in a subgame associated with player 1 having chosen c_1), but it is not a subgame perfect Nash equilibrium since player 1 is not choosing the optimal action at the information set associated with player 1 having chosen c_1 and player 2 having chosen b_2.

Solution Exercise 8

8. Consider the extensive form game below. The top number at a terminal node is player 1's payoff, the second number is player 2's payoff, the third number is player 3's payoff, and the bottom number is player 4's payoff.



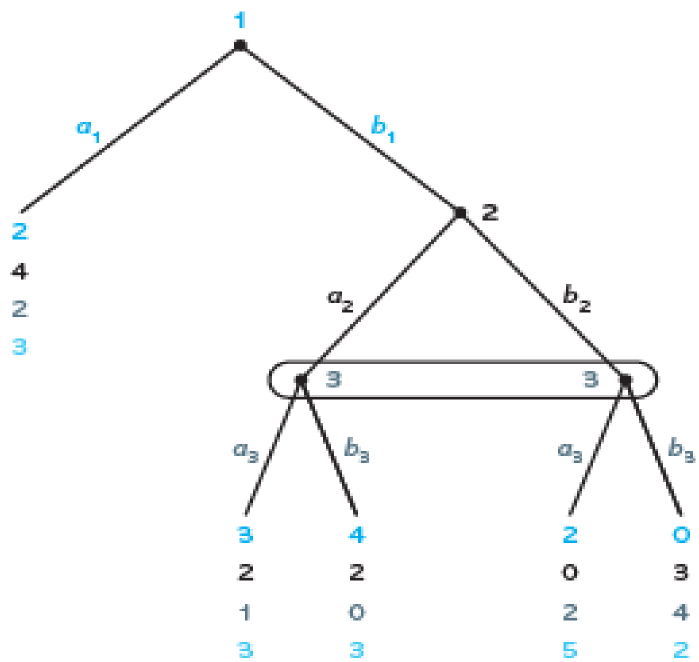
a. Derive the strategy set for each player or, alternatively, state a representative strategy for a player.

ANSWER: Player 1 has only one information set, so his strategy set is made up of a_1 and b_1. Player 2 has two information sets, so a strategy is a pair of actions. Let the first action be associated with when player 1 chose a_1 and the second action be associated with when player 1 chose b_1. Player 2's strategy set is then

$(a_2/a_2, a_2/b_2, b_2/a_2, b_2/b_2)$. Player 3's strategy set is made up of a_3 and b_3. Finally, a strategy for player 4 is a 4-tuple of actions since she has four information sets. A strategy for player 4 takes the form *w/x/y/z*, where *w* is played when player 1 chose b_1, player 2 chose a_2, and player 3 chose a_3; *x* is played when player 1 chose b_1, player 2 chose a_2, and player 3 chose b_3; *y* is played when player 1 chose b_1, player 2 chose b_2, and player 3 chose a_3; and *z* is played when player 1 chose b_1, player 2 chose b_2, and player 3 chose b_3. *w*, *x*, *y*, and *z* take the value a_4 or b_4.

b. Derive all subgame perfect Nash equilibria.

ANSWER: Consider the subgame associated with player 1 having chosen a_1. It has a unique Nash equilibrium in which player 2 chooses b_2. For the subgame in which player 1 chose b_1, player 2 chose a_2, and player 3 chose b_3, it has a unique Nash equilibrium of a_4. For the subgame in which player 1 chose b_1, player 2 chose a_2, and player 3 chose b_3, it has a unique Nash equilibrium of b_4. For the subgame in which player 1 chose b_1, player 2 chose b_2, and player 3 chose a_3, it has a unique Nash equilibrium of b_4. And for the subgame in which player 1 chose b_1, player 2 chose b_2, and player 3 chose b_3, it has a unique Nash equilibrium of a_4. Substituting these payoffs for these five subgames, we have the figure below.



Consider the subgame reached by player 1 choosing b_1. Its strategic form is shown in the figure below and it has two Nash equilibria: (a_2, a_3) and (b_2, b_3).

Player 3

Player 2		a_3	b_3
	a_2	2,1	2,0
	b_2	0,2	3,4

Substituting this subgame with the payoffs associated with Nash equilibrium (a_2, a_3), there is a unique Nash equilibrium for the game faced by player 1 at the initial decision node which has player 1 choose b_1. This gives us a subgame perfect Nash equilibrium, $(b_1, b_2/a_2, a_3, a_4/b_4/b_4/a_4)$. Now go back to the subgame reached after player 1 chose b_1 and use instead the Nash equilibrium (b_2, b_3). The Nash equilibrium for the game faced by player41 at the initial decision node is a_1. This gives us a second subgame perfect Nash equilibrium of $(a_1, b_2/b_2, b_3, a_4/b_4/b_4/a_4)$.

Exercise 9

ANSWER: The strategic form game of the subgame associated with the path *no pitchout* → *no suicide squeeze* is:

		LaRussa	
		Suicide squeeze	*No suicide squeeze*
Torre	*Pitchout*	9,1	4,6
	No pitchout	3,7	5,5

This game has no pure strategy Nash equilibrium. Suppose p is the probability that LaRussa chooses *suicide squeeze*, and q is the probability that Torre chooses *pitchout* in a mixed-strategy Nash equilibrium. Then, for Torre to be indifferent between *pitchout* and *no pitchout* it must be the case that

$$p \times 9 + (1 - p) \times 4 = p \times 3 + (1 - p) \times 5 \Rightarrow p = \frac{1}{7}.$$

For LaRussa to be indifferent between *suicide squeeze* and *no suicide squeeze*, it must be the case that

$$q \times 1 + (1 - q) \times 7 = q \times 6 + (1 - q) \times 5 \Rightarrow q = \frac{2}{7}.$$

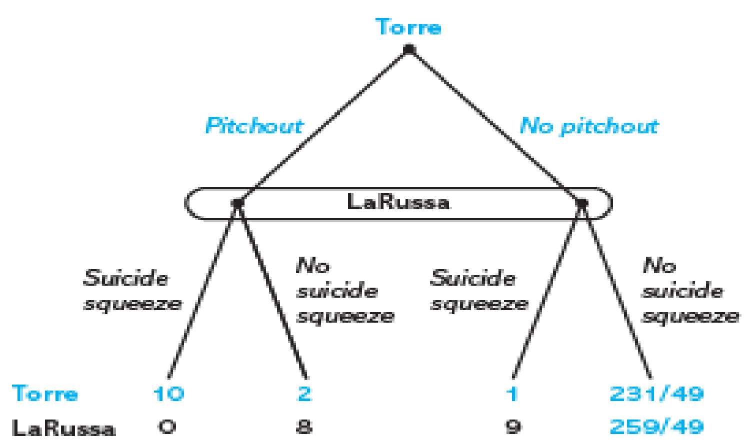
Hence, a mixed-strategy Nash equilibrium has Torre choose *pitchout* with probability $\frac{2}{7}$ and LaRussa choose *suicide squeeze* with probability $\frac{1}{7}$. Torre's expected payoff in this equilibrium is

$$\frac{2}{49} \times 9 + \frac{12}{49} \times 4 + \frac{5}{49} \times 3 + \frac{30}{49} \times 5 = \frac{231}{49}.$$

For LaRussa, it is:

$$\frac{2}{49} \times 1 + \frac{12}{49} \times 6 + \frac{5}{49} \times 7 + \frac{30}{49} \times 5 = \frac{259}{49}.$$

Replacing the subgame with those payoffs, the subgame is as shown in the figure below.



The strategic form is shown in the figure below.

		LaRussa	
		Suicide squeeze	*No suicide squeeze*
Torre	*Pitchout*	10,0	2,6
	No pitchout	1,9	231/49,259/49

This game does not have a pure strategy Nash equilibrium. Now let a be the probability that LaRussa chooses *suicide squeeze,* and b be the probability that Torre chooses *pitchout*. Then, for Torre to be indifferent between *pitchout* and *no pitchout*, we need

$$a \times 10 + (1 - a) \times 2 = a \times 1 + (1 - a) \times \frac{231}{49} \Rightarrow a = \frac{19}{63}.$$

For LaRussa to be indifferent between *suicide squeeze* and *no suicide squeeze*, we need

$$b \times 0 + (1 - b) \times 9 = b \times 8 + (1 - b) \times \frac{259}{49} \Rightarrow b = \frac{13}{41}.$$

Hence, at a mixed-strategy Nash equilibrium, Torre chooses *pitchout* with probability $\frac{13}{41}$ at his first decision node and with probability $\frac{2}{7}$ at his second decision node. Meanwhile, LaRussa chooses *suicide squeeze* with probability $\frac{19}{63}$ at his first decision node and with probability $\frac{1}{7}$ at his second decision node.

2. Consider the sequential-move game depicted in Figure 1. The game describes Apple's decision to develop the new iPhone with radically new software which allows for faster applications (apps). These apps are, however, still not developed by app developers. If Apple does not develop the new iPhone, then all companies make zero profit in this emerging market, If, instead, the new iPhone is developed, then company 1 (the leader in the app industry) gets to decide whether to develop apps that are compatible with the new iPhone's software. Upon observing company 1's decision, the followers (firm 2 and 3) simultaneously decide whether to develop apps (D) or not develop (ND), Find all SPNE in this sequential-move game.

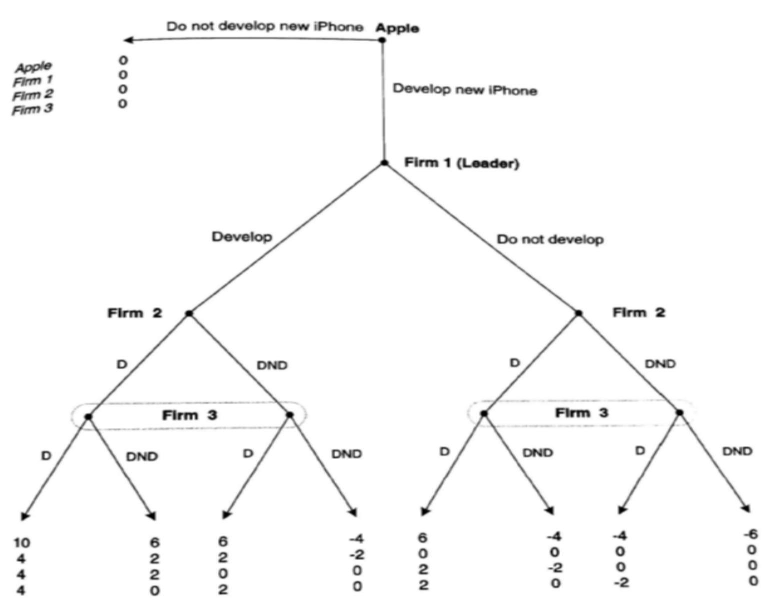


Figure 1

Solution

Company 1 develops. Consider the subgame between firms 2 and 3 which initiates after Apple develops the new iPhone and company 1 develops an application (in the left-hand side of the game tree of Fig. 1). Since that subgame describes that firm 2 and 3 simultaneously choose whether or not to develop apps, we must represent it using its normal form in order to find the NEs of this subgame; as we do in the payoff matrix of Fig. 1a.

		Firm 3	
		Develop	Do not develop
Firm 2	Develop	**4,4**	2,0
	Do not develop	0,2	0,0

Fig. 1a Smallest proper subgame (after firm 1 develops)

We can identify the best responses for each player (as usual, the payoffs associated to those best responses are underlined in the payoff matrix of Fig. 1a). In particular,

$$BR_2(D,D) = D \text{ and } BR_2(D,ND) = D \quad \text{for firm 2;}$$

and and similarly for firm 3,

$$BR_3(D,D) = D \text{ and } BR_3(D,ND) = D \ .$$

Hence, Develop is a dominant strategy for each company, so there is a unique Nash equilibrium (Develop, Develop) in this subgame.

Company 1 does not develop. Next, consider the subgame associated with Apple having developed the new iPhone but company 1 not developing an application (depicted in the right-hand side of the game tree in Fig. 1). Since the subgame played between firms 2 and 3 is simultaneous, we represent it using its normal form in Fig. 1b.

		Firm 3	
		Develop	Do not develop
Firm 2	Develop	**2,2**	-2,0
	Do not develop	0,-2	**0,0**

Fig. 1b Smallest proper subgame (after firm 1 does not develop)

This subgame has two Nash equilibria: (Develop, Develop) and (Do not develop, Do not develop).
Note that, in our following discussion, we will have to separately analyze the case in which outcome (D, D) emerges as the NE of this subgame, and that in which (ND, ND) arises.

Company 1—Case I. Let us move up the tree to the subgame initiated by Apple having developed the new iPhone. At this point, company 1 (the industry leader) has to decide whether or not to develop an application. Suppose that the Nash equilibrium for the subgame in which company 1 does not develop an application is (Develop, Develop). Replacing the two final subgames with the Nash equilibrium payoffs we found in our previous discussion, the situation is as depicted in the tree of Fig. 1c. In particular, if firm 1 develops an application, then outcome (D, D) which entails payoffs (10, 4, 4, 4), as depicted in the terminal node at the bottom left-hand side of Fig. 1c. If, in contrast, firm 1 does not develop an application for the new iPhone, then outcome (D, D) arises, yielding payoffs of (6, 0, 2, 2), as depicted in the bottom right–hand corner of Fig. 1c.

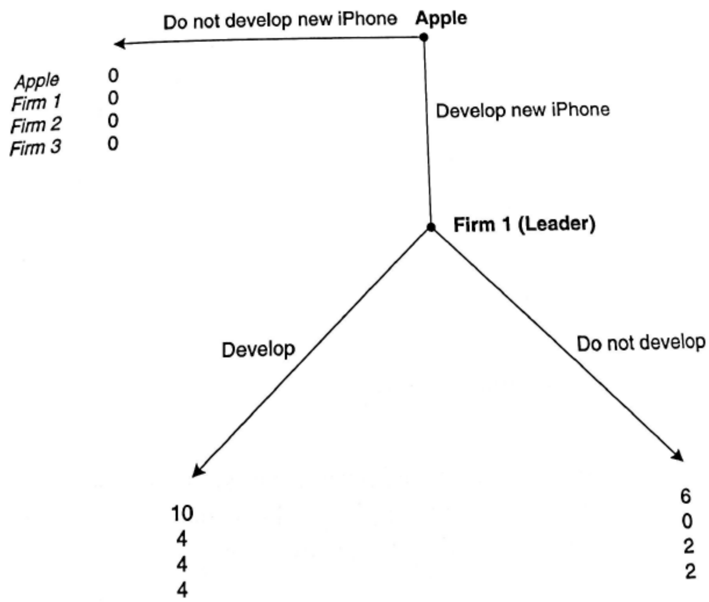


Fig. 1c Extensive-form subgame (Case I)

We can now analyze firm 1's decision. If company 1 develops an application, then its payoff is 4, while its payoff is only 0 (since it anticipates the followers developing apps) from not doing so. Hence, company 1 chooses Develop.

Company 1—Case II. Now suppose the Nash equilibrium of the game that arises after firm 1 does not develop an app has neither firm 2 or 3 developing an app. Replacing the two final subgames with the Nash equilibrium payoffs we found in our previous discussion, the situation is as depicted in Fig. 1d. Specifically, if firm 1 develops, outcome (D, D) emerges, which entails payoffs (10, 4, 4, 4); while if firm 1 does not develop firm 2 and 3 respond not developing apps either, ultimately yielding a payoff vector of (-6, 0, 0, 0). In this setting, if firm 1 develops an application, its payoff is 4; while its payoff is only 0 from not doing so. Hence, firm 1 chooses Develop.

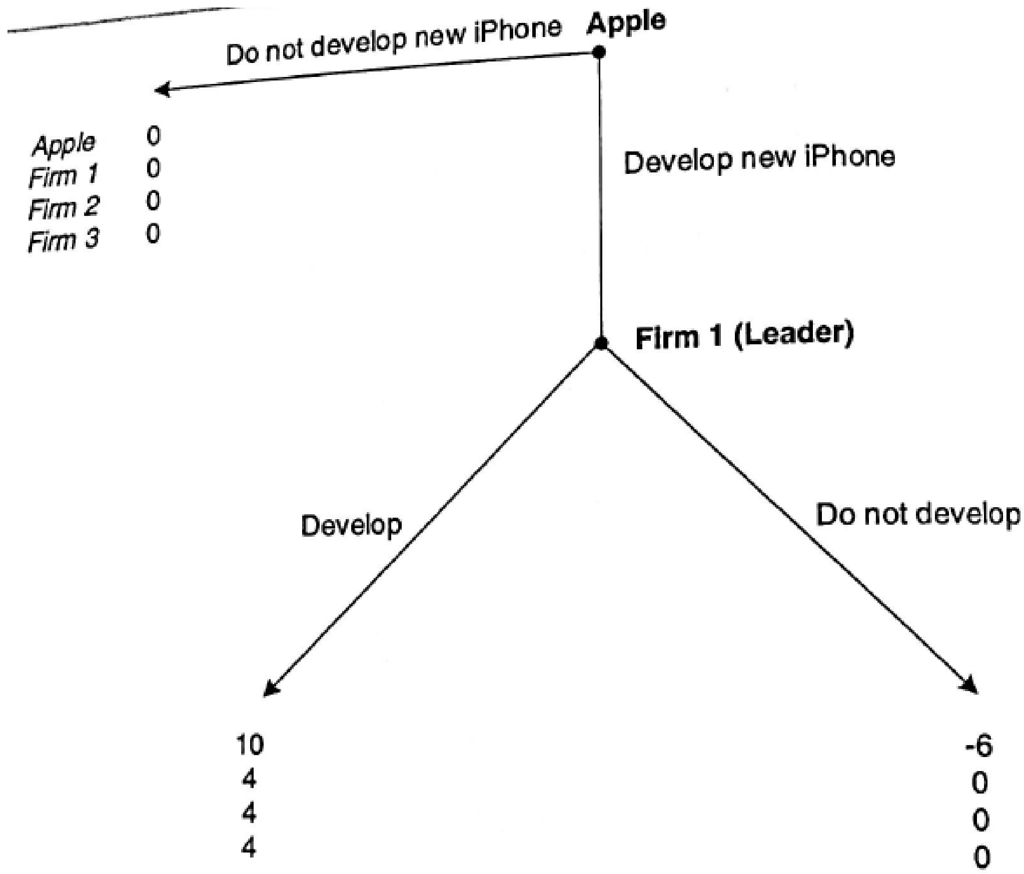


Fig. 1d Extensive-form subgame (Case II)

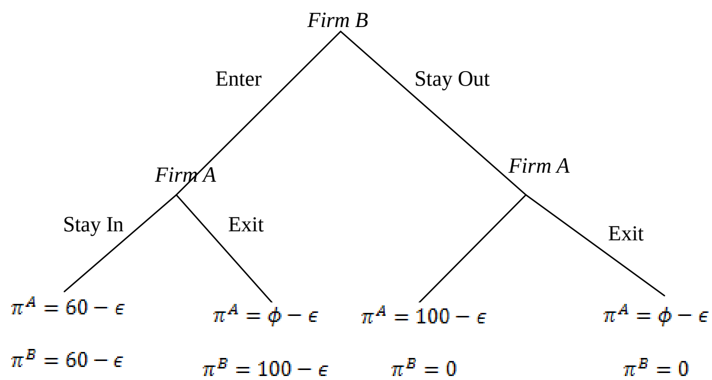
Thus, regardless of which Nash equilibrium is used in the subgame initiated after firm 1 chooses Do not develop (in the right-hand side of the game in Figs. 1c and 1d), firm 1 (the leader) optimally chooses to Develop.

First mover(Apple). Operating by backward induction, we now consider the first mover in this game (Apple). If Apple chooses to develop the new iPhone, then, as previously derived, firm 1 develops an application and this induces all followers 2 and 3 to do so as well. Hence, Apple's payoff is 10 from introducing the new iPhone. It is then optimal for Apple to develop the new iPhone, since its payoffs from so doing, 10, is larger than from not developing it, 0. Intuitively, since Apple anticipates all app developers will react introducing new apps, it finds the initial introduction of the iPhone to be very profitable. We can then identify two subgame perfect Nash equilibria (where a strategy for firm 2, as well as for firm 3, specifies a response to firm 1 choosing Develop and a response to company 1 choosing Do not develop):

(Develop iPhone, Develop, Develop/Develop, Develop/Develop), and
(Develop iPhone, Develop, Develop/Do not develop, Develop/Do not develop).

Note that both SPNE result in the same equilibrium path, whereby, first, Apple introduces the new iPhone, the industry leader (firm 1) subsequently chooses to develop applications for the new iPhone, and finally firms 2 and 3 (observing firm 1's apps development) simultaneously decide to develop apps as well.

3. Let us analyze an entry-exit two-stage game in which firm A is the incumbent and firm B is a potential entrant. In stage I, firm B chooses whether to enter into A's market or whether to stay out. The cost of entry is denoted by ϵ. In the second stage, firm A decides whether to stay in the market or exit.



The game tree reveals that firm A can recover some of its sunk entry cost by selling its capital for the price ϕ, where $0 \leq \phi \leq \epsilon$.

a. Obtain the subgame-perfect equilibrium strategies of both firms assuming that $\epsilon < 60$. Prove your answer.

b. Answer the above assuming that $60 < \phi \leq \epsilon < 100$

Solution

a) *Since* $\epsilon < 60$, *it must be that* $\phi < 60$. *Hence,* $60 - \epsilon > \phi - \epsilon$. *Therefore, Firm A's SPE strategy is*

$$S_A = \begin{cases} \textit{stay} \text{ if } S_B = \textit{enter} \text{ (because } 60 - \epsilon > \phi - \epsilon) \\ \textit{stay} \text{ if } S_B = \textit{out} \text{ (because } 100 - \epsilon > \phi - \epsilon) \end{cases}$$

The SPE strategy of firm B (first mover) is $s_B = \textit{enter}$ *(because* $60 - \epsilon > 0$*).*

b) *Now,* $60 - \epsilon < \phi - \epsilon$. *Therefore, Firm A's SPE strategy is*

$$S_A = \begin{cases} \textit{exit} \text{ if } S_B = \textit{enter} \text{ (because } 60 - \epsilon < \phi - \epsilon) \\ \textit{stay} \text{ if } S_B = \textit{out} \text{ (because } 100 - \epsilon > \phi - \epsilon) \end{cases}$$

The SPE strategy of firm B (first mover) is $s_B = \textit{enter}$ *(because* $100 - \epsilon > 0$*).*

4. Consider a leader and a follower in a Stackelberg game of quantity competition. Firms face an inverse demand curve $p(Q) = 1 - Q$, where $Q = q_L + q_F$, denotes aggregate output. The leader faces a constant marginal cost $c_L > 0$ while the follower's marginal cost is $c_F > 0$, where $1 > c_F > c_L$, indicating that the leader has a cost advantage.

(a) Find the follower's best response function.

(b) Determine each firm's output strategy in the SPNE of this sequential-move game.

(c) Under which conditions on c_L can you guarantee that both firms produce strictly positive output levels?

Solution

Part (a) The follower observes the leader's output level, , and chooses its own production, , to solve:

$$\max(1 - q_L - q_F)q_F - c_F q_F$$

Taking first order conditions with respect to q_F yields

$$1 - q_L - 2q_F - c_F = 0$$

and solving for we obtain the follower's best response function

$$q_F(q_L) = \frac{1 - c_F}{2} - \frac{1}{2}q_L$$

which, as usual, is decreasing in the follower's costs, i.e., the vertical intercept decreases in c_F, indicating that, graphically, the best-response function experiences a downward shift as c_F increases. In addition, the follower's best-response function decreases in the leader's output decision (as indicated by the negative slope, $-1/2$).

Part (b) The leader anticipates that the follower will respond with best response function $q_F(q_L) = \frac{1-c_F}{2} - \frac{1}{2}q_L$, and plugs it into the leader's own profit maximization problem, as follows

$$\max(1 - q_L - (\frac{1 - c_F}{2} - \frac{1}{2}q_L))q_L - c_L q_L$$

which simplifies into

$$\max \frac{1}{2}[(1 + c_F) - q_L]q_L - c_L q_L$$

Taking first order conditions with respect to yields

$$\frac{1}{2}(1 + c_F) - q_L - c_L = 0$$

and solving for we find the leader's equilibrium output level

$$q_L^* = \frac{1 + c_F - 2c_L}{2}$$

which, thus, implies a follower's equilibrium output of

$$\begin{aligned} q_F &= \frac{1 - c_F}{2} - \frac{1}{2}(\frac{1 + c_F - 2c_L}{2}) \\ q_F^* &= \frac{2(1 - c_F) - (1 + c_F - 2c_L)}{4} = \frac{1 - 3c_F + 2c_L}{4} \end{aligned}$$

Note that the equilibrium output of every firm $i = L, F$ is decreasing in its own cost, c_i, and increasing in its rival's cost, c_j, where $j \neq i$. Hence, the SPNE of the game is

$$(q_L^*, q_F(q_L)) = (\frac{1 + c_F - 2c_L}{2}, \frac{1 - c_F}{2} - \frac{1}{2}q_L)$$

which allows the follower to optimally respond to both the equilibrium output level from the leader, q_L^*, but also to off-the-equilibrium production decisions $q_L \neq q_L^*$.

Part (c) The follower (which operates under a cost disadvantage), produces a positive output level in equilibrium, i.e., $q_F^* > 0$, if and only if

$$\frac{1 - 3c_F + 2c_L}{4} > 0$$

which, solving for c_L, yields

$$c_L > \frac{3c_F - 1}{2} \equiv C_A$$

Similarly, the leader produces a positive output level, $q_L^* > 0$, if and only if

$$\frac{1 + c_F - 2c_L}{2} > 0$$

or, solving for c_L,

$$c_L < \frac{1 + c_F}{2} \equiv C_B$$

Figure 2 depicts cutoffs C_A (for the follower) and C_B (for the leader), in the $(c_L, c_F\ )$—quadrant. (Note that we focus on points below the 45°-line, since the leader experiences a cost advantage relative to the follower, i.e. $c_F > c_L$,

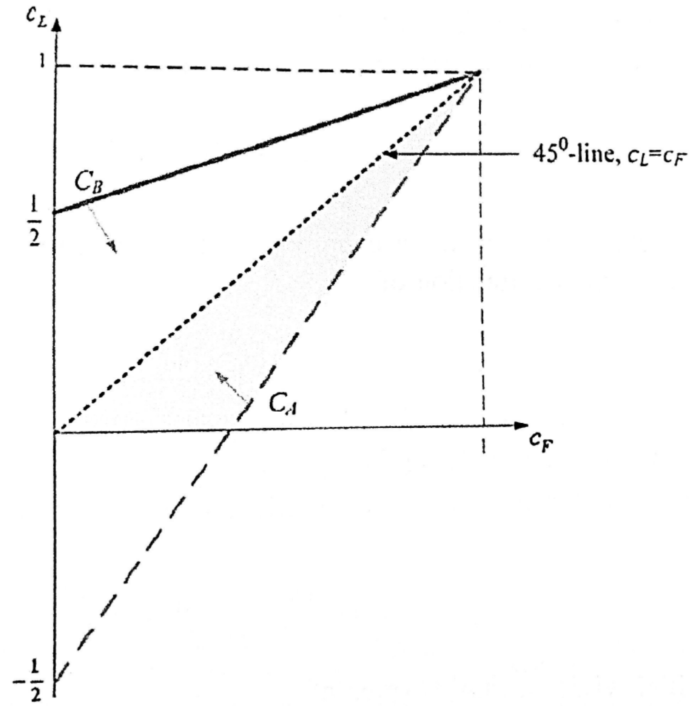


Fig. 2 Region of cost pairs for which both firms produce positive output

First, note that cutoff C_B is not binding since it lies above the 45°-line. Intuitively, the leader produces a positive output level for all (c_L, c_F)—pairs in the admissible region of cost pairs (below the 45°-line). However, cutoff C_A restricts the of cost pairs below the 45°-line to only that above cutoff C_A . Hence, in the shaded area of the figure, both firms produce a strictly positive output in equilibrium.