EconS 424 – Spring 2021

Midterm exam #1

Exercise 1 – Applying IDSDS in three-player games

Consider the following anti-coordination game in Figure 1 played by three potential entrants seeking to enter into a new industry, such as the development of software applications for smartphones. Every firm (labeled as A, B, and C) has the option of entering or staying out (i.e., remain in the industry they have been traditionally operating, e.g., software for personal computers). The normal form game in figure 1 depicts the market share that each firm obtains, as a function of the entering decision of its rivals. Firms simultaneously and independently choose whether or not to enter. As usual in simultaneous-move games with three players, the triplet of payoffs describes the payoff for the row player (firm A) first, for the column player (firm B) second, and for the matrix player (firm C) third.

a) Find the set of strategy profiles that survive the iterative deletion of strictly dominated strategies (IDSDS).
b) Is the equilibrium you found using this solution concept unique?

Firm C: Enter

Firm A \ Firm B	Enter	Stay Out
Enter	14,24,32	8,30,27
Stay Out	30,16,24	13,12,50

Firm C: Stay Out

Firm A \ Firm B	Enter	Stay Out
Enter	16,26,30	31,16,24
Stay Out	31,23,14	14,26,32

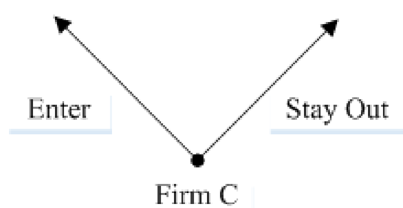

Figure 1. Normal-form representation of a three-players game

Answer:

We can start by looking at the payoffs for firm C (the matrix player). [Recall that the application of IDSDS is insensitive to the deletion order. Thus, we can start deleting strictly dominated strategies for the row, column or matrix player, and still reach the same equilibrium result.] In particular, let us compare the third payoff of every cell across both matrices. Figure 1.22 provides you a visual illustration of how to do this pairwise comparison across matrices.

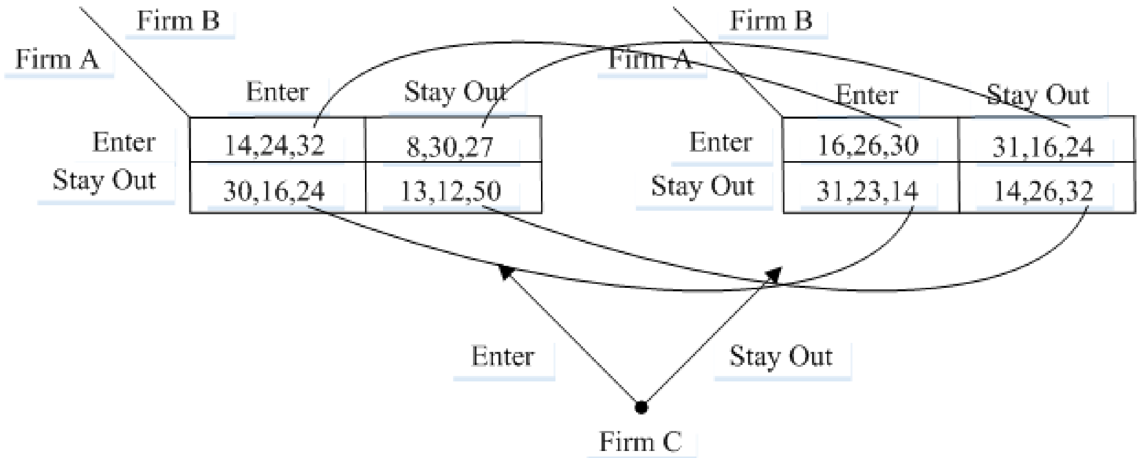

Figure 1.22. Pairwise payoff comparison for firm C

We find that for firm C (matrix player), entering strictly dominates staying out, i.e., $u_C(s_A,s_B,E)>u_C(s_A,s_B,O)$ for any strategy of firm A, s_A, and firm B, s_B, 32>30, 27>24, 24>14 and 50>32 in the pairwise payoff comparison depicted in figure 1.22 This allows us to delete the right-hand side matrix (corresponding to firm C choosing to stay out) since it would not be selected by firm C. We can, hence, focus on the left-hand matrix alone (where firm C chooses to enter), which we reproduce in figure 1.23.

Firm A \ Firm B	Enter	Stay Out
Enter	14,24,32	8,30,27
Stay Out	30,16,24	13,12,50

Figure 1.23. Reduced Normal-form game

We can now check that entering is strictly dominated for the row player (firm A), i.e., $u_A(E,s_B,E)<u_A(O,s_B,E)$ for any strategy of firm B, s_B, once we take into account that firm C selects its strictly dominant strategy of entering. Specifically, firm A prefers to stay out both when firm B enters (in the left-hand column, since 30>14), and when firm B stays out (in the right-hand column, since 13>8). In other words, regardless of firm B's decision, firm A prefers to stay out. This allows us to delete the top row from the above matrix, since the strategy "Enter" would never be used by firm A, which leaves us with a single row and two columns, as illustrated in figure 1.24.

Firm A \ Firm B	Enter	Stay Out
Stay Out	30,16,24	13,12,50

Figure 1.24. Reduced Normal-form game

Once we have done that, the game becomes an individual-decision making problem, since only one player (firm B) must select whether to enter or stay out. Since entering yields a payoff of 16 to firm B, while staying out only entails 12, firm B chooses to enter, given that it regards staying out as a strictly dominated strategy, i.e., $u_B(O,E,E) > u_B(O,O,E)$ where we fix the strategies of the other two firms at their strictly dominant strategies: staying out for firm A and entering for firm C. We can thus delete the column corresponding to staying out in the above matrix, as depicted in figure 1.25.

Firm A \ Firm B	Enter
Stay Out	30, 16, 24

Figure 1.25

As a result, the only surviving cell (strategy profile) that survives the application of the iterative deletion of strictly dominated strategies (IDSDS) is that corresponding to (Stay Out, Enter, Enter), which predicts that firm A stays out, while both firms B and C choose to enter.

Exercise 2 – Tournaments

Several strategic settings can be modeled as a tournament, whereby the probability of winning a certain prize not only depends on how much effort you exert, but also on how much effort other participants in the tournament exert. For instance, wars between countries, or R&D competitions between different firms in order to develop a new product, not only depend on a participant's own effort, but on the effort put by its competitors. Let's analyze equilibrium behavior in these settings. Consider that the benefit that firm 1 obtains from being the first company to launch a new drug is $36 million. However, the probability of winning this R&D competition against its rival (i.e., being the first to launch the drug) is $\frac{x_1}{x_1+x_2}$, which it increases with this firm's own expenditure on R&D, x_1, relative to total expenditure, $x_1 + x_2$. Intuitively, this suggests that, while spending more than its rival, i.e., $x_1 > x_2$, increases firm 1's chances of being the winner, the fact that $x_1 > x_2$ does not guarantee that firm 1 will be the winner. That is, there is still some randomness as to which firm will be the first to develop the new drug, e.g., a firm can spend more resources than its rival but be "unlucky" because its laboratory exploits a few weeks before being able to develop the drug. For simplicity, assume that firms' expenditure cannot exceed 25, i.e., $x_i \in [0, 25]$.The cost is simply x_i, so firm 1's profit function is

$$\pi_1(x_1, x_2) = 36\left(\frac{x_1}{x_1 + x_2}\right) - x_1$$

and there is an analogous profit function for country 2:

$$\pi_2(x_1, x_2) = 36\left(\frac{x_2}{x_1 + x_2}\right) - x_2$$

You can easily check that these profit functions are concave in a firm's own expenditure, i.e., $\frac{\partial^2 \pi_i(x_{i,} x_j)}{\partial x_i^2} \leq 0$ for every firm $i=\{1, 2\}$ where $j \neq i$. Intuitively, this indicates that, while profits increase in the firm's R&D, the first million dollar is more profitable than the 10th million dollar, e.g., the innovation process is more exhausted.

a. Find each firm's best-response function.

b. Find a symmetric Nash equilibrium, i.e., $x_1^* = x_2^* = x^*$.

Answer:

Firm 1's optimal expenditure is the value of x_1 for which the first derivative of its profit function equals zero. That is,

$$\frac{\partial \pi_1(x_1, x_2)}{\partial x_1} = 36\left[\frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2}\right] - 1 = 0$$

Rearranging, we find

$$36\left[\frac{x_2}{(x_1 + x_2)^2}\right] - 1 = 0$$

$$36x_2 = (x_1 + x_2)^2$$

and further rearranging

$$6\sqrt{x_2} = x_1 + x_2$$

Solving for x_1, we obtain firm 1's best response function

$$x_1(x_2) = 6\sqrt{x_2} - x_2$$

Figure 2.26.depicts firm 1's best response function, $x_1(x_2) = 6\sqrt{x_2} - x_2$ as a function of its rival's expenditure, x_2 in the horizontal axis for the admissible set $x_2 \in [0, 25]$.

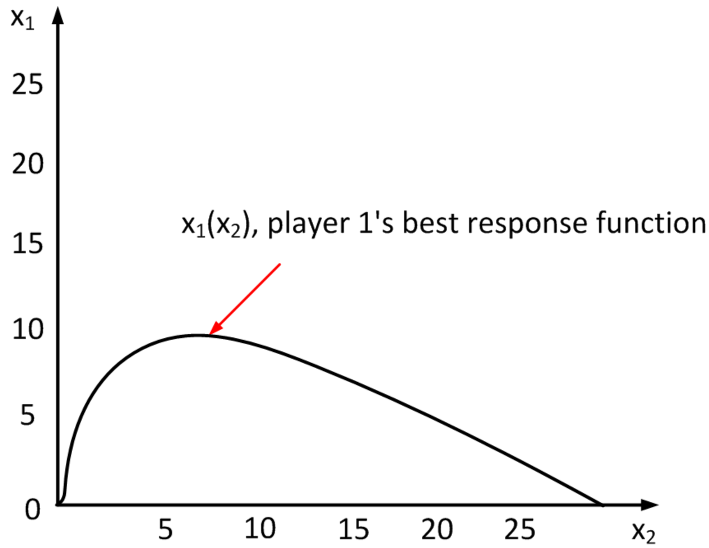

Figure 2.26. Firm 1's Best Response Function

It is straightforward to show that, for all values of $x_2 \in [0, 25]$, firm 1's best response also lies in the admissible set $x_1 \in [0, 25]$. In particular, the maximum of BRF_1 occurs at x_2=9 since

$$\frac{\partial BR_1(x_2)}{\partial x_2} = \frac{\partial [6\sqrt{x_2} - x_2]}{\partial x_2} = 3(x_2)^{-\frac{1}{2}} - 1$$

Hence, the point at which this best response function reaches its maximum is that in which its derivative is zero, i.e., $3(x_2)^{-\frac{1}{2}} - 1 = 0$, which yields a value of $x_2 = 9$. At this point, firm 1's best response function informs us that firm 1 optimally spends $6\sqrt{9} - 9 = 9$. Finally, note that the best response function is concave in its rival expenditure, x_2, since

$$\frac{\partial^2 BR_1(x_2)}{\partial x_2} = -\frac{3}{2}(x_2)^{-\frac{3}{2}} < 0.$$

By symmetry, firm 2's best response function is $x_2(x_1) = 6\sqrt{x_1} - x_1$.

b. Find a symmetric Nash equilibrium, i.e., $x_1^* = x_2^* = x^*$

A symmetric Nash equilibrium is an expenditure level, denoted x^*, such that a firm finds it optimal to spend x^* when the other firm spends x^*. It is then a solution to

$$x^* = 6\sqrt{x^*} - x^*$$

Rearranging, we obtain $2x^* = 6\sqrt{x^*}$, and solving for x^*, we find $x^* = 9$. Hence, the unique symmetric Nash equilibrium has each firm spending 9. As figure 2.27 depicts, the points at which the best response function of player 1 and 2 cross each other occur at the 45-degree line (so the equilibrium is symmetric). In particular, those points are the origin, i.e., (0,0), but this case is uninteresting since it implies that no firm spends money on R&D, and (9,9).

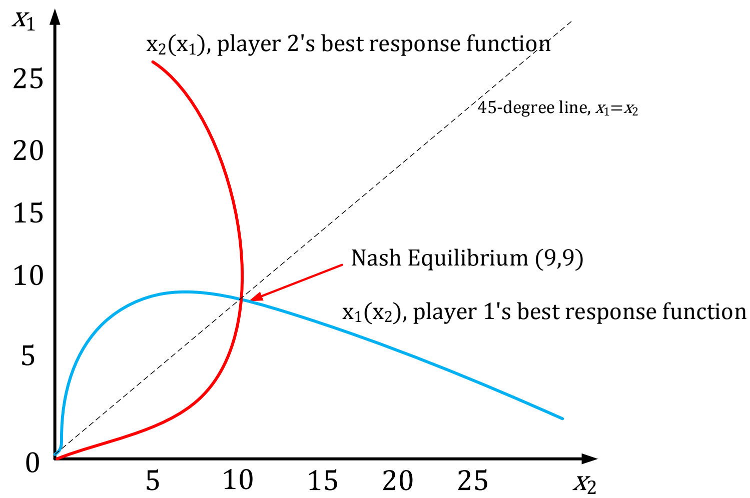

Figure 2.27. Best Response Functions. Nash-Equilibrium

Exercise 3 - Game of Chicken

Consider the game of Chicken (depicted in figure 2), in which two players driving their cars against each other must decide whether or not to swerve.

		Player 2 Straight	Swerve
Player 1	Straight	0,0	3,1
	Swerve	1,3	2,2

Figure 2. Game of Chicken

a. Is there any strictly dominated strategy for Player 1? And for Player 2?
b. What are the best responses for Player 1? And for Player 2?
c. Can you find any pure strategy Nash Equilibrium (psNE) in this game?
d. Find the mixed strategy Nash Equilibrium (msNE) of the game. *Hint:* denote by p the probability that Player 1 chooses Straight and by $(1-p)$ the probability that he chooses to Swerve. Similarly,

let q denote the probability that Player 2 chooses Straight and $(1-q)$ the probability that she chooses to Swerve.

e. Show you result of part (d) by graphically representing every player i's best response function $BRF_i(s_j)$, where $s_j = \{Swerve, Straight\}$ is the strategy selected by player $j \neq i$.

Answer:

a. No, there are no strictly dominated strategies for any player. In particular, Player 1 prefers to play Straight when Player 2 plays Swerve getting a payoff of 3 instead of 2 if he chooses to Swerve. However, he prefers to Swerve when Player 2 plays Straight, getting a payoff of 1 by swerving instead of a payoff of 0 if he chooses to go Straight. Hence, there is no strictly dominated strategy for player 1, i. e., a strategy that Player 1 would never use regardless of what strategy his opponent selects. The incentives of Player 2 are symmetric.

b. *Player 1.* If player 2 chooses straight (fixing our attention in the left-hand column), player 1's best response is to Swerve, since his payoff from doing so, 1, is larger than that from choosing Straight. Hence,

$$BR_1(Straight) = Swerve, \text{ obtaining a payoff of 1}$$

If player 2 instead chooses to Swerve (in the right-hand column), player 1's best response is Straight, since his payoff from Straight, 3, exceeds that from Swerve, 2. That is,

$$BR_1(Swerve) = Straight, \text{ obtaining a payoff of 3}$$

Player 2. A similar argument applies to player 2's best responses; since players are symmetric in their payoffs. Hence,

$$BR_2(Straight) = Swerve, \text{ obtaining a payoff of 1}$$
$$BR_2(Swerve) = Straight, \text{ obtaining a payoff of 3}$$

Therefore, we have an Anti-Coordination game, where each player responds by choosing exactly the opposite strategy of his/her competitor.

c. Note that in strategy profiles (Swerve, Straight) and (Straight, Swerve), there is a mutual best response by both players. To see this more graphically, the matrix in figure 3.2 underlines the payoffs corresponding to each player's selection of his/her best response. For instance, $BR_1(Straight) = Swerve$ implies that, when Player 2 chooses Straight (in the left-hand column), Player 1 responds Swerving (in the bottom left-hand side cell), obtaining a payoff of 1; while when Player 2 chooses Swerve (in the right-hand column) $BR_1(Swerve) = Straight$, and thus Player 1 obtains a payoff of 3 (top right-hand side cell). A similar argument applies to Player 2. There are, hence, two cells where the payoffs of all players have been underlined i.e., mutual best responses: (Swerve, Straight) and (Straight, Swerve). These are the two pure strategy Nash equilibria of the Chicken game.

		Player 2	
		Straight	Swerve
Player 1	Straight	0,0	<u>3</u>,<u>1</u>
	Swerve	<u>1</u>,<u>3</u>	2,2

Figure 3.2. Nash equilibria in the Game of Chicken

d. *Player 1.* Let q denote the probability that Player 2 chooses Straight, and $(1-q)$ the probability that she chooses Swerve as depicted in figure 3.2.

			Player 2	
			Straight	Swerve
			q	$1 - q$
Player 1	Straight	P	0,0	3,1
	Swerve	$1 - p$	1,3	2,2

Figure 3.2. Nash equilibria in the Game of Chicken

In this context, the expected value that Player 1 obtains from playing Straight (in the top row) is:

$$EU_1(Straight) = 0q + 3(1-q)$$

since, fixing our attention in the top row, player 1 gets a payoff of zero when player 2 chooses Straight (which occurs with probability q) or a payoff of 3 when player 2 selects Swerve (which happens with the remaining probability 1-q). If, instead, player 1 chooses to Swerve (in the bottom row), his expected utility becomes

$$EU_1(Swerve) = 1q + 2(1-q)$$

since player 1 obtains a payoff of 1 when player 2 selects Straight (which occurs with the probability q) or a payoff of 2 when player 2 chooses Swerve (which happens with probability 1-q). Player 1 must be indifferent between choosing Straight or Swerve. Otherwise, he would not be randomizing, since one pure strategy would generate a larger expected payoff than the other, leading Player 1 to select such a pure strategy. We therefore need that

$$EU_1(Straight) = EU_1(Swerve)$$

$$0q + 3(1-q) = 1q + 2(1-q)$$

rearranging and solving for probability q yields

$$3 - 3q = q + 2 - 2q \quad \Rightarrow \quad q = \frac{1}{2}$$

That is, Player 2 must be choosing Straight with 50% probability, i.e., $q = \frac{1}{2}$, since otherwise Player 1 would not be indifferent between Straight and Swerve.

Player 2. When he chooses Straight (in the left-hand column), he obtains an expected utility of

$$EU_2(sraight) = 0p + 3(1-p)$$

since he can get a payoff of zero when player 1 chooses Straight (which occurs with probability p, as depicted in figure 3.3), or a payoff of when player 1 selects to Swerve (which happens with probability 1-

p). When, instead, player 2 chooses Swerve (directing our attention in the right-hand column), he obtains an expected utility of

$$EU_2\left(swerve\right)=1p+2\left(1-p\right)$$

since his payoff is 1 if player 1 chooses Straight (which occurs with probability p) or a payoff of 2 if player 1 selects to Swerve (which happens with probability 1-p). Therefore, player 2 is indifferent between choosing Straight or Swerve when

$$EU_2(Straight) = EU_2(Swerve)$$

$$p0 + 3(1-p) = 1p + 2(1-p)$$

Solving for probability p in the above indifference condition, we obtain

$$1 = 2p \quad \Rightarrow \quad p = \frac{1}{2}$$

Hence, the mixed strategy Nash equilibrium of the Chicken game prescribes that every player randomizes between driving Straight and Swerve half of the time. More formally, the mixed strategy Nash equilibrium (msNE) is given by

$$msNE = \left\{\left(\frac{1}{2}Straight, \frac{1}{2}Swerve\right), \left(\frac{1}{2}Straight, \frac{1}{2}Swerve\right)\right\}$$

e. We first draw the thresholds which specify the mixed strategy Nash equilibrium (msNE): $p = 1/2$ and $q = 1/2$. Then, we draw the two pure strategy Nash equilibria found in question (c):

- One psNE in which Player 1 plays Straight and Player 2 plays Swerve (which in probability terms means $p = 1$ and $q = 0$), which is graphically depicted at the lower right-hand corner of figure 3.4 (in the southeast); and

- Another psNE in which Player 1 plays Swerve and Player 2 plays Straight (which in probability terms means $p = 0$ and $q = 1$), which is graphically depicted at the upper left-hand corner of figure 3.4 (in the northwest).

Player 1's best response function. Once we have drawn these pure strategy equilibria, notice that from player 1's best response function, $BR_1(q)$, we know that, if $q < \frac{1}{2}$, then $EU_1(Straight) > EU_1(Swerve)$, thus implying that Player 1 plays Straight using pure strategies, i.e., $p = 1$. Intuitively, when player 1 knows that player 2 is rarely selecting Straight, i.e., $q < \frac{1}{2}$, his best response is to play Straight, i.e., $p = 1$ for all $q < \frac{1}{2}$, as illustrated in the vertical segment of $BR_1(q)$ in the right-hand side of figure 3.4. In contrast, $q > \frac{1}{2}$ entails $EU_1(Straight) < EU_1(Swerve)$, and therefore $p = 0$. In this case, player 2 is likely playing Straight, leading player 1 to respond with Swerve, i.e., p=0, as depicted in the vertical segment of $BR_1(q)$ that overlaps the vertical axis in the left-hand side of figure 3.4.

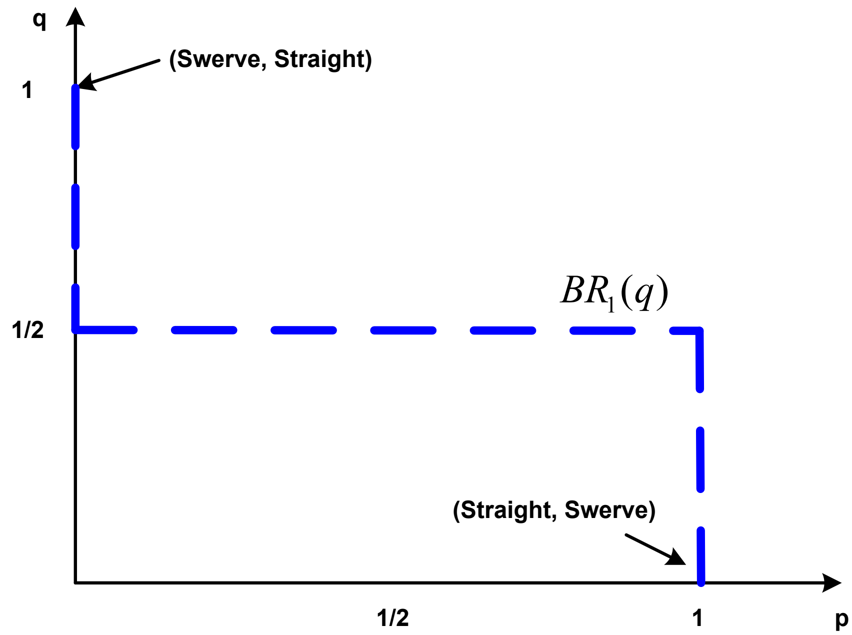

Figure 3.4. Best Response Function of Player 1

Player 2's best response function. A similar analysis applies to player 2's best response function, $BR_2(p)$, depicted in figure 3.5: (1) when $p < \frac{1}{2}$, player 2's expected utility comparison satisfies $EU_2(Straight) > EU_2(Swerve)$, thus implying $q = 1$, as depicted in the horizontal segment of $BR_2(p)$ at the top right-hand side of figure 3.5; and (2) when $p > \frac{1}{2}$, $EU_2(Straight) < EU_2(Swerve)$, ultimately leading to $q = 0$, as illustrated by the horizontal segment of $BR_2(p)$ that overlaps the horizontal axis of the figure.

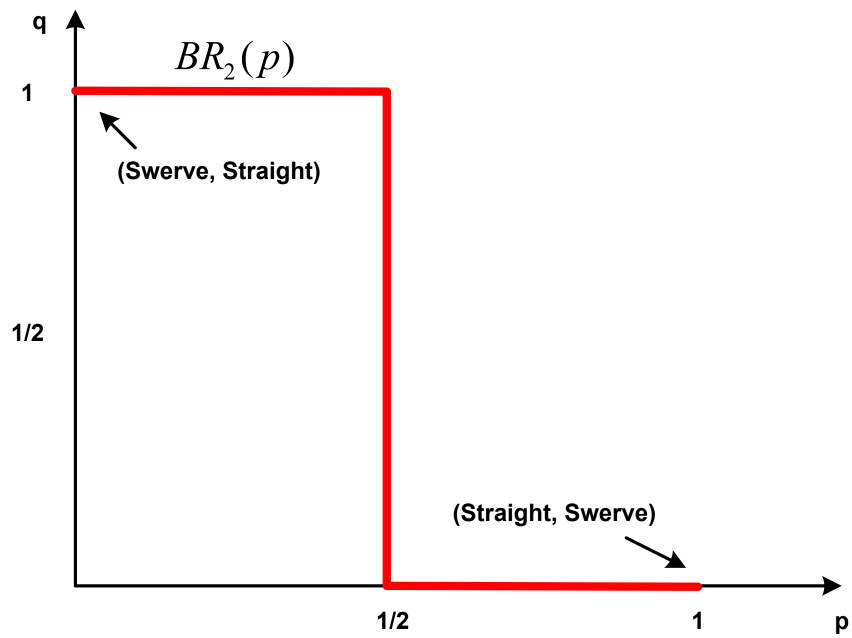

Figure 3.5. Best Response Function of Player 2

Superimposing both players' best response functions, we obtain figure 3.6, which depicts the two psNE of this game (southeast and northwest corners), as well as the msNE (strictly interior point, where players are randomizing their strategies).

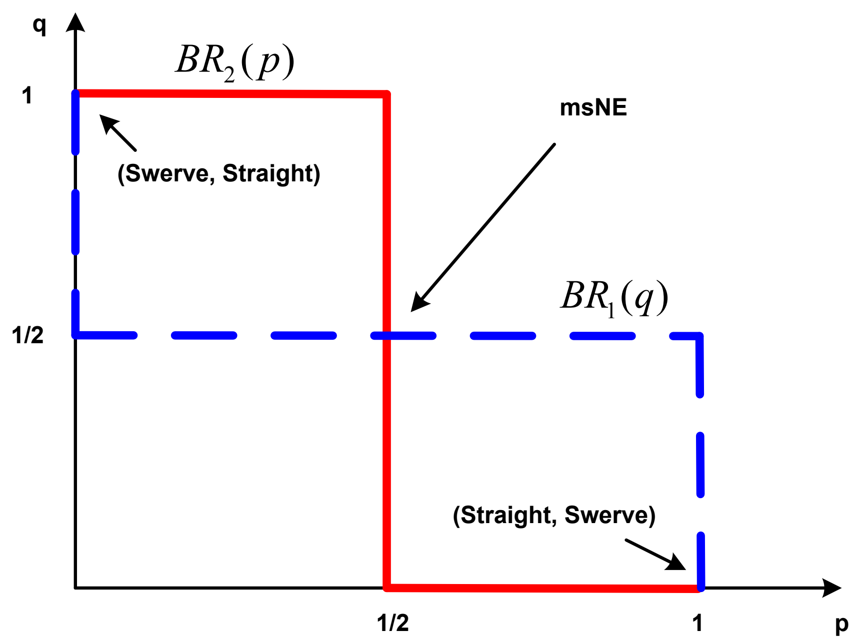

Figure 3.6. Best Response Functions, psNE and msNE

Exercise 4 – BONUS EXERCISE - Cournot game [15 points]

Consider two neighboring wineries in fierce competition over the production of their specialty wine (where their grapes come from the same vineyard, so the wines are exactly the same), One owned by Jaclyn (J), the other by Roey (R). Each winery produces their wine the same way and have the symmetric total cost function $TC_i(q_i)=3+0.5q_i$ where i=J,R. Inverse market demand for wine is $p=50-2(q_J + q_R)$.

a. Cournot competition. Write down the profit-maximization problem for each firm if they compete in quantities (à la Cournot).
b. Using the profit-maximization problems you wrote in part (a), find each firm's best response function. Interpret.
c. Using the best response functions you found in part (b), what is each winery's equilibrium output and price?

Answer:

a)Jaclyn's profit-maximization problem is

$$\max \pi_J=[50-2(q_J + q_R)]\, q_J - (3+0.5\, q_J)$$

Roey's maximization-problem is

$\max \pi_R = [50-2(q_J + q_R)]\, q_R - (3+0.5\, q_R)$

b)Using the profit-maximization problems you wrote in part (a), find each firm's best response function. Interpret.

Jaclyn's best response function. If we differentiate Jaclyn's profit with respect to its output q_J, we obtain

$$\left(\frac{\partial \pi_J}{\partial q_J}\right) = 50 - 4\, q_J - 2q_R - 0.5 = 0$$

rearranging, we have $4\, q_J = 49.5 - 2q_R$. Solving for q_J, we obtain Jaclyn's best response function

$$q_J\,(q_R) = 12.375 - (1/2)q_R$$

which originates at a vertical intercept of 12.375 and decreases at a rate of (1/2) for every unit of Roey's output.

Roey's best response function. Differentiating Roey's profit with respect to its output q_R, we obtain

$$\left(\frac{\partial \pi_R}{\partial q_R}\right) = 50 - 4q_R - 2q_J - 0.5 = 0$$

rearranging, we have $4q_R = 49.5 - 2q_J$. Solving for q_R, we obtain Roey's best response function

$$q_R(q_J) = 12.375 - (1/2)q_J$$

which is symmetric to Jaclyn's best response function. This comes at no surprise since both wineries face the same inverse demand function and total cost function.

c)Using the best response functions you found in part (b), what is each winery's equilibrium output and price?

In a symmetric equilibrium, both winneries produce the same output level, $q_J = q_R = q$, which means that the above best response function becomes

$$q = 12.375 - (1/2)q.$$

Rearranging, we find $(3/2)q = 12.375$, and solving for q yields equilibrium output

$$q^* = 12.375(2/3) = 8.25 \text{ units.}$$

Plugging these quantities into the inverse demand, we get the equilibrium price

$$p^* = 50 - 2(8.25+8.25) = \$17$$

and equilibrium profits are

$$\pi^* = p^*q^* - (3+0.5q^*) = (17\times 8.25) - (3+0.5\times 8.25) = 140.25 - 7.125 = \$133.125$$