ECONS 424 – STRATEGY AND GAME THEORY

HOMEWORK #2 – ANSWER KEY

Exercise 1: Harrington

One of the critical moments early on in the Lord of the Rings trilogy is the meeting in Rivendale to decide who should take the ring to Mordor. Gimli the dwarf won't hear of an elf doing it, while Legolas (who is an elf) feels similarly about Gimli. Boromir (who is a man) is opposed to either of them taking charge of the ring. He is also held in contempt, for it was his ancestor who, when given the opportunity to destroy the ring millennia ago, chose to keep it instead. And then there is Frodo the hobbit, who has the weakest desire to take the ring, that someone must throw it into the fires of Mordor. In modeling this scenario as a game, assume there are four players: Boromir, Frodo, Gimli, and Legolas. (There were more of course, including Aragorn and Elrond, but lets keep it simple.) Each of them has a preference ordering, shown in the following table, as to who should take on the task of carrying the ring.

Preference Rankings For The Lord of the Rings:

Person	First	Second	Third	Fourth	Fifth
Boromir	Boromir	Frodo	No one	Legolas	Gimli
Gimli	Gimli	Frodo	No one	Boromir	Legolas
Legolas	Legolas	Frodo	No one	Gimli	Boromir
Frodo	Legolas	Gimli	Boromir	Frodo	No one

Of the three non-hobbits, each prefers to have himself take on the task. Other than themselves and Frodo, each would prefer that no one take the ring. As for Frodo, he doesn't really want to do it and prefers to do so only if no one else will. The game is one in which all players simultaneously make a choice among the four people. Only if they all agree – a unanimity voting rule is put in place – is someone selected; otherwise, no one takes on this epic task. Find all symmetric Nash equilibria.

Answer: There are four symmetric strategy profiles and thus four candidates for symmetric Nash equilibrium. Note that if a person fails to vote for the person that everyone else votes for, then no one takes on the task. Thus, at a symmetric strategy profile, an individual player's choice is always between the person who the others are voting for and no one. Consider the symmetric strategy profile in which all vote for Boromir. This strategy is optimal for both Boromir and Frodo, as each would rather that Boromir take on the task than no one do so. This is clearly not a Nash equilibrium however, as both Legolas and Gimli would prefer to vote for someone else, and the result would be that no one takes the ring to Mordor. By a similar argument, it is not a Nash equilibrium for all to vote for Gimli, or for all to vote for Legolas. Now consider all voting for Frodo. Since each person prefers that Frodo do it than that no one do it, each player's strategy is optimal. The unique symmetric Nash equilibrium is then for all to vote for Frodo.

1) $u_i(\text{Boromir}_i, \text{Boromir}_{-i}) > u_i(\text{No one}_i, \text{Boromir}_{-i})$ holds for i= Boromir and i=Frodo, but does not hold for for Gimli and Legolas

2) $u_i(\text{Gimli}_i, \text{Gimli}_{-i} > u_i(\text{No one}_i, \text{Gimli}_{-i})$ holds for i= Gimli and i= Frodo, but does not hold for Legolas and Boromir
3) $u_i(\text{Legolas}_i, \text{Legolas}_{-i}) > u_i(\text{No one}_i, \text{Legolas}_{-i})$ holds for i = Legolas and i = Frodo, but does not hold for Boromir and Gimli.
4) $u_i(\text{Frodo}_{i,}, \text{Frodo}_{-I}, \text{Frodo}_{-i}) > u_i(\text{No one}_i, \text{Frodo}_{-i})$ holds for all players i.

For this problem, everyone has to agree on who takes on the task. So you have to go down the list of people and find the person where everyone agrees.

To start we look at if Boromir was chosen, him and Frodo would agree, but Legolas and Gimli would choose no one.

Next we move to Gimli, and he and Frodo would choose Gimli, but Legolas and Boromir would choose no one.

Legolas and Frodo would choose Legolas, but Boromir and Gimli would choose no one.

Finally we get to Frodo and everyone would choose Frodo before they pick no one.

One of the critical moments early on in the Lord of the Rings

Exercise 9: Harrington

Find all the Nash equilibria for the three-player game.

Player 3: A

Player 2

Player 1

	X	Y	Z
a	1,1,0	2,0,0	2,0,0
b	3,2,1	1,2,3	0,1,2
c	2,0,0	0,2,3	3,1,1

Player 3: B

Player 2

Player 1

	X	Y	Z
a	2,0,0	0,0,1	2,1,2
b	1,2,0	1,2,1	1,2,1
c	0,1,2	2,2,1	2,1,0

Player 3: C

Player 2

Player 1

	X	Y	Z
a	2,0,0	0,1,2	0,1,2
b	0,1,1	1,2,1	0,1,2
c	3,1,2	0,1,2	1,1,2

To find the Nash equilibrium, we have to first look at the best response from player 3. When we start to do the underlining we get the following:

Player 3: A

Player 2

Player 1

	X	Y	Z
a	1,1,0	2,0,0	2,0,0
b	3,2,1	1,2,3	0,1,2
c	2,0,0	0,2,3	3,1,1

Player 3: B

Player 2

Player 1

	X	Y	Z
a	2,0,0	0,0,1	2,1,2
b	1,2,0	1,2,1	1,2,1
c	0,1,2	2,2,1	2,1,0

Player 3: C

Player 2

Player 1

	X	Y	Z
a	2,0,0	0,1,2	0,1,2
b	0,1,1	1,2,1	0,1,2
c	3,1,2	0,1,2	1,1,2

Next we do the same for player one and get the following:

Player 3: A

Player 2

Player 1

	X	Y	Z
a	1,1,0	2,0,0	2,0,0
b	3,2,1	1,2,3	0,1,2
c	2,0,0	0,2,3	3,1,1

Player 3: B

Player 2

Player 1

	X	Y	Z
a	2,0,0	0,0,1	2,1,2
b	1,2,0	1,2,1	1,2,1
c	0,1,2	2,2,1	2,1,0

Player 3: C

Player 2

Player 1

	X	Y	Z
a	2,0,0	0,1,2	0,1,2
b	0,1,1	1,2,1	0,1,2
c	3,1,2	0,1,2	1,1,2

Finally we look at the best responses for player 2:

Player 3: A

Player 2

Player 1

	X	Y	Z
a	1,1,0	2,0,0	2,0,0
b	3,2,1	1,2,3	0,1,2
c	2,0,0	0,2,3	3,1,1

Player 3: B

Player 2

Player 1

	X	Y	Z
a	2,0,0	0,0,1	2,1,2
b	1,2,0	1,2,1	1,2,1
c	0,1,2	2,2,1	2,1,0

Player 3: C

Player 2

Player 1

	X	Y	Z
a	2,0,0	0,1,2	0,1,2
b	0,1,1	1,2,1	0,1,2
c	3,1,2	0,1,2	1,1,2

The Nash equilibria are then (b, x, A), (a, z, B), (c,x,C) and (c, z, C)

Question 5: Harrington, Chapter 5 – Exercise 5

a) With 2 diners there will be a 3x3 payoff matrix:

P1\P2	Pasta	Salmon	Filet
Pasta			
Salmon			
Filet			

To find the payoffs for (Pasta, Pasta) for example, we have:
Value(21) - Total Cost(14+14)/2 = Payoff(7)

This is the same for both players as this game is symmetric.
Again, the payoff for (Pasta, Salmon) for player 1 is:
21-(14+21)/2 = 21-17.5 = 3.5

For P2 his payoff is 26-(14+21)/2 = 26-17.5 = 8.5

Repeating these calculations we find,

P1\P2	Pasta	Salmon	Filet
Pasta	7,7	3.5, 8.5	-1, 7
Salmon	8.5, 3.5	5, 5	.5, 3.5
Filet	7, -1	3.5, 0.5	-1, -1

Using best response for each player, it is simple to show that (salmon, salmon) is the Nash equilibrium.

b. Suppose there are four diners (n=4). What will they order (at a Nash equilibrium)?

Answer: Note that a diner cannot influence what others order and must pay 25% of the price of the ordered meals. All that a diner an influence is her own order. The key property to note is that whatever she orders, she pays only 25% of the price with the remaining 75% being paid by the other three diners. Once recognizing that this is the actual cost to her, not the price on the menu, a diner should choose the meal that maximizes her surplus. Taking all this into account, Table Sol 5.5.2 shows the costs faced by a diner. For example, a diner who orders the pasta dish pays only 25% of the menu price, which is $3.50. We observe that each diner orders the filet mignon because it really only costs them $7.50, and the surplus is maximized with that order. The unique Nash equilibrium is then that all four diners order the steak. Hence, each gets a meal he or she values at $29, but ends up paying $30!

Dish	Value	Actual Cost	Surplus
Pasta	$21.00	$3.50	$17.50
Salmon	$26.00	$5.25	$20.75
Filet	$29.00	$7.50	$21.50

Harrington, Chapter 5 – Exercise 6

To find all of the Nash equilibria we need to find where firms do not wish to enter or leave.

Number of Firms	Gross Profit Per Firm	Net Profit Per Firm
1	1000	700
2	400	100
3	250	-5
4	150	-150
5	100	-200

Because each company's "bottom line" is Net Profit, this is the relevant information to analyze. With one firm, it's evident to other firms that they may make money by entry, so an additional firm enters. At two firms though, the next firm entering would lose money, so it's clear that 2 firms is the Nash equilibrium. Testing the other possibilities, starting at # of firms=5, there is incentive to exit until only two firms remain. So the only equilibrium is at two firms.

Harrington, Chapter 5 – Exercise 11

Answer: This game has a unique Nash equilibrium, which is for all tho take the course: xi = 1 for all i. Suppose all the students other than i take the course: xi =1 for all j ≠i. Suppose i = 1. Since a2 + z > a1, then student 1's ranking without having taken the course is no higher than second (it could have been lower if a3 + z > a1.) Thus, her payoff from not taking the course is no higher than b(n-2). If she takes the course, she will be ranked first, since a1 + z > ai +z for all j≠i. Hence, her payoff is b(n-1) – c. Next, note that b(n-1)- c > b(n-2), which is equivalent to b > c, which is true by assumption. It has then been shown that the payoff to student 1 from taking the course exceeds her payoff from not taking the course, given everyone else takes the course. It is then optimal for student 1to take the course. The gain in score and ranking from taking the prep course exceeds the cost to student 1.

Next, consider student i, where 2 ≤ i ≤ n -2. The analysis is similar to that for student 1. If student i takes the course-given all other students take the course- her payoff is b(n-i)-c. Her payoff is b(n-i)-c. Her payoff from not taking the course is no higher than b(n-i-1). Since b(n-i)-c > b(n-i-1), she prefers to take the course. Finally, consider student i , where i = n – 1, or i = n. Her payoff from not taking the course is b(n-n) = 0 as she is ranked last. Her payoff from taking the course is b(n-(n-1)) – c = b – c > 0, so she prefers to take the course. This completes the proof that all the students taking the course is a Nash equilibrium. One can show that this is the unique Nash equilibrium.

Exercise 2 – Cournot competition with 3 firms

Consider three firms competing *a la* Cournot, in a market with inverse demand function $P(Q) = 1 - Q$, and production costs normalized to zero.

a. Find the psNE of the game when firms simultaneously and independently choose quantities. Determine the equilibrium profit level for each firm.
b. Consider now that two (out of three) firms merge, and thus choose their output decision in order to maximize their joint profits. Find the psNE in this game for the merged firms and the unmerged firms. Identify the equilibrium profits for each firm, and compare them with your results pre-merger in part (a)
c. Consider now that all three firms merge. Find their profit maximizing output and profits, comparing them with your results in (a) and (b).

Answer:

a. The profits for firm *i* are

$$\pi_i = (1 - Q)q_i = \left(1 - q_i - q_j - q_k\right)q_i$$

Taking first order conditions with respect to q_i, we obtain:

$$1 - 2q_i - q_j - q_k = 0 \quad \Longrightarrow \quad q_i(q_j, q_k) = \frac{1 - q_j - q_k}{2}$$

And in a symmetric Nash equilibrium in which all firms are producing the same output, i. e., $q_i = q_j = q_k = q$, we find $q_i = \frac{1}{4}$ for every firm i.

Hence, equilibrium prices are $p = 1 - Q = 1 - 3\frac{1}{4} = \frac{1}{4}$

And, therefore, profits for every firm are (recall that there are no production costs)

$$\pi_i = pq_i = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

b. There are only two firms in the market now: the merge of firms 1 and 2, and the (unmerged) firm 3. The profits for either of these *two* firms are:

$$\pi_i = (1 - Q)q_i = (1 - q_i - q_j)q_i$$

And taking FOCs with respect to q_i, we obtain:

$$1 - 2q_i - q_j = 0 \quad \Longrightarrow \quad q_i(q_j) = \frac{1 - q_j}{2}$$

and in a symmetric Nash equilibrium in which all firms are producing the same output, i. e., $q_i = q_j = q$, we find $q_i = \frac{1}{3}$ for every firm i.

Hence, equilibrium prices are $p = 1 - Q = 1 - 2\frac{1}{3} = \frac{1}{3}$

And, therefore, profits for every firm are

$$\pi_i = pq_i = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

This implies that

- ✓ Firms 1 and 2 obtain profits of $\frac{\frac{1}{9}}{2} = \frac{1}{18}$ after the merger, which are lower than the pre-merger profits of $\frac{1}{16}$
- ✓ Firm 3 obtains profits of $\frac{1}{9}$, which exceed its pre-merger profits of $\frac{1}{16}$

Intuition: the merged firms internalize part of price reduction that an increase in their aggregate production entails, i. e., they consider the profit loss that the increase in production by one of the firm participating in the merger entails on the other firm that joined the merger. As a consequence, the merged firms reduce their individual production relative to pre-merger levels (in the standard Cournot competition analyzed in part a). However, the unmerged Firm 3 does not take into these price effects, and must responds to a lower output level from both of its competitors by increasing its own production. Ultimately, the firms that merged obtain a lower profit than before the merger, while the merged firm earns a larger profit. This result is often referred as the "merger paradox".

c. If all firms merge, they form a cartel, acting as a monopolist. [Note that this is only true when they all merge, not when only two of them merge, as we examined in the previous section]. When they all merge their joint profits are

$$\pi_i = (1 - Q)Q = Q - Q^2$$

Taking first order conditions with respect to Q, we obtain

$$1 - 2Q = 0 \quad \Longrightarrow \quad Q = \frac{1}{2}$$

Which implies that equilibrium price is

$$p = 1 - Q = 1 - \frac{1}{2} = \frac{1}{2}$$

And equilibrium profits are

$$\pi = pQ = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Therefore, the individual profits of every firm participating in the merger are $\frac{\frac{1}{4}}{3} = \frac{1}{12}$, which are clearly higher than their profits pre-merger (when all firms compete as Cournot oligopolists) of $\frac{1}{16}$.

Exercise 3 – Mixed strategy Nash equilibrium with *N* players

a) The normal form representation of the game for *n*=2 players is given below.

		Player 2	
Player 1		*X*	*Y*
	X	$\underline{3},\underline{3}$	$\underline{4},\underline{3}$
	Y	$\underline{3},\underline{4}$	2,2

There are three pure strategy Nash equilibria in this game, (X,X), (X,Y) and (Y,X).

b) When introducing *n*=3 players, the normal form representation of the game is:

- First, if Player 3 chooses X,

		Player 2	
Player 1		*X*	*Y*
	X	0,0,0	$\underline{3},\underline{3},\underline{3}$
	Y	$\underline{3},\underline{3},\underline{3}$	$2,2,\underline{4}$

- And if Player 3 chooses Y,

		Player 2	
Player 1		*X*	*Y*
	X	$\underline{3},\underline{3},\underline{3}$	$\underline{4},2,2$
	Y	$2,\underline{4},2$	1,1,1

Hence, the pure strategy Nash equilibria of the game with n=3 players are (X,Y,X), (Y,X,X) and (X,X,Y).

c) If every player is choosing X with probability p and Y with probability 1-p, the expected utility that player 1 obtains by playing X is:

- $EU_1(X)=p^2 0+p(1-p)3+ (1-p)p3+(1-p)^2 4=p(1-p)6+4(1-p)^2$

And player 1's utility from playing Y is:

- $EU_1(Y)=p^2 3+p(1-p)2+(1-p)p2+(1-p)^2 1=3p^2+4(1-p)p+(1-p)^2$

Player 1 is indifferent between choosing strategy X and Y for values of p such that $EU_1(X)= EU_1(Y)$. That is,

- $p(1-p)6+4(1-p)^2=3p^2+4(1-p)p+(1-p)^2$, and simplifying, $2p^2+4p-3=0$

- Solving for p, we find that either p= -1-$\sqrt{2.5}$ <0 (which cannot be a solution to our problem, since $p \in [0,1]$, or p= -1+$\sqrt{2.5}$ =0.58, which is the solution to our problem.

Hence, every player in this game randomizes between X and Y (using mixed strategies) assigning probability p=0.58 to strategy X, and 1-p=0.42 to strategy Y.