

# Mixed strategy Nash equilibrium

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Week 5

## Looking back...

- So far we have been able to find the NE of a relatively large class of games with complete information:
  - Games with two or several ( $n > 2$ ) players.
  - Games where players select among discrete or continuous actions.
- But, can we assure that all complete information games where players select their actions simultaneously have a NE?
  - We couldn't find a NE for the matching pennies game!! (Next slide)
  - We will be able to claim existence of a NE if we allow players to randomize their actions.

## Remembering the "matching pennies" game...

- Recall that this was an example of an anti-coordination game:

		$P_2$	
		Head	Tail
$P_1$	Head	<u>1</u> , -1	-1, <u>1</u>
	Tail	-1, <u>1</u>	<u>1</u> , -1

Indeed, there is no strategy pair in which players select a particular action 100% of the times.

- We need to allow players to randomize their choices.

## Another example

- Here we have another example of an anti-coordination game with no psNE:

Surprise!

*Police Officer*

Street Corner

Park

*Drug Dealer*

Street Corner

Park

Street Corner	<u>80</u> , 20	0, <u>100</u>
Park	10, <u>90</u>	<u>60</u> , 40

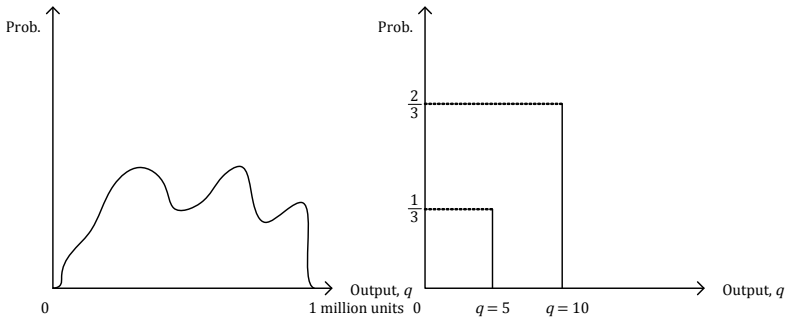
- We need to allow players randomize their choices (i.e., to play mixed strategies).

# Mixed strategy Nash equilibrium

- Harrington: Chapter 7, Watson: Chapter 11.
- First, note that if a player plays more than one strategy with strictly positive probability, then he must be indifferent between the strategies he plays with strictly positive probability.
- **Notation:** "non-degenerate" mixed strategies denotes a set of strategies that a player plays with strictly positive probability.
  - Whereas "degenerate" mixed strategy is just a pure strategy (because of degenerate probability distribution concentrates all its probability weight at a single point).

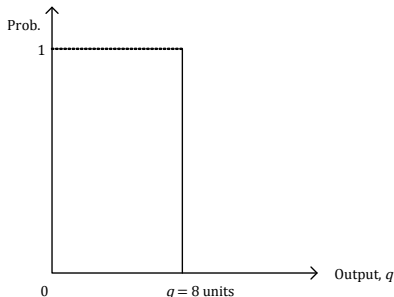
# Degenerate Probability Distributions

- Example of non-degenerate probability distributions



# Degenerate Probability Distributions

- Example of a degenerate probability distribution



- The player (e.g., firm) puts all probability weight (100%) on only one of its possible actions:  $q = 8$ .

- **Definition of msNE:**

- Consider a strategy profile  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$  where  $\sigma_i$  is a mixed strategy for player  $i$ .  $\sigma$  is a msNE if and only if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \text{ for all } s'_i \in S_i \text{ and for all } i$$

- That is,  $\sigma_i$  is a best response of player  $i$  to the strategy profile  $\sigma_{-i}$  of the other  $N - 1$  players,  $\sigma_i = BR_i(\sigma_{-i})$ .



- Notice that we wrote  $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\mathbf{s}'_i, \sigma_{-i})$  instead of  $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$ .
- **Why?** If a player was using  $\sigma'_i$ , then he would be indifferent between all pure strategies to which  $\sigma'_i$  puts a positive probability, for example  $\hat{s}_i$  and  $\check{s}_i$ .
  - That is why it suffices to check that no player has a profitable pure-strategy deviation.

## Example 1: Matching pennies

- **Matching pennies**

		<i>Player 2</i>	
		$q$	$1 - q$
<i>Player 1</i>	$p$ Heads	1, -1	-1, 1
	$1 - p$ Tails	-1, 1	1, -1

- **Two alternative interpretations of players' randomization:**

- If player 1 is using a mixed strategy, it must be that he is indifferent between Heads and Tails
- Alternatively, if player 1 is indifferent between Heads and Tails, it must be that player 2 mixes with such probability  $q$  such that player 1 is made indifferent between Heads and Tails:

$$EU_1(H) = EU_1(T) \iff 1q + (1 - q)(-1) = (-1)q + 1(1 - q)$$

## Matching pennies

- **Matching pennies** (example of a normal form game with no psNE):

		<i>Player 2</i>	
		$q$	$1 - q$
<i>Player 1</i>	$p$ Heads	1, -1	-1, 1
	$1 - p$ Tails	-1, 1	1, -1

- Solving for the EU comparison, we obtain

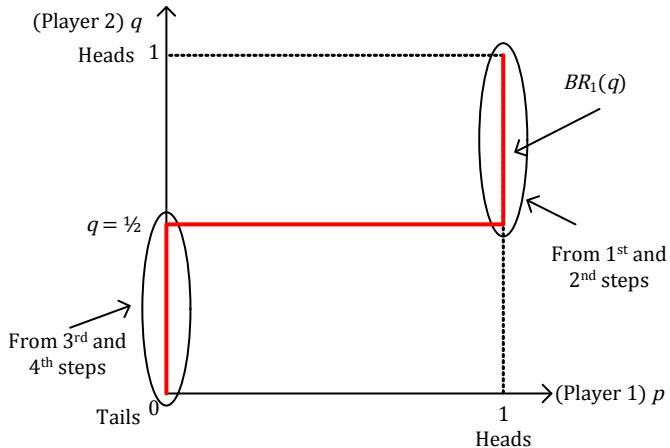
$$EU_1(H) = EU_1(T) \iff 1q + (1 - q)(-1) = (-1)q + 1(1 - q)$$

$$q = \frac{1}{2} \longrightarrow \text{Graphical Interpretation}$$

# Matching pennies

- How to interpret this cutoff of  $q = \frac{1}{2}$  graphically?
  - 1 We know that if  $q > \frac{1}{2}$ , then player 2 is very likely playing Heads. Then, player 1 prefers to play Heads as well ( $p = 1$ ).
    - Alternatively, note that  $q > \frac{1}{2}$  implies  $EU_1(H) > EU_1(T)$ .
  - 2 Go to the figure on the next slide, and draw  $p = 1$  for every  $q > \frac{1}{2}$ .
  - 3 If  $q < \frac{1}{2}$ , player 2 is likely playing Tails. Then, player 1 prefers to play Tails as well ( $p = 0$ ).
  - 4 Graphically, draw  $p = 0$  for every  $q < \frac{1}{2}$ .

# Matching pennies



## Matching pennies

- Similarly, if player 2 is using a mixed strategy, it must be that he is indifferent between Heads and Tails:

$$EU_2(H) = EU_2(T)$$

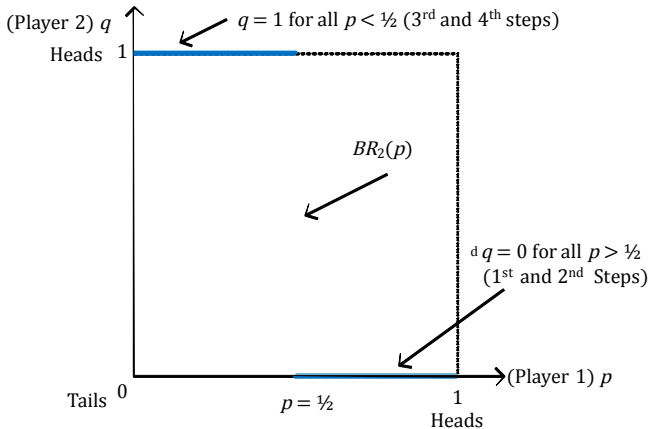
$$(-1)p + 1(1 - p) = 1p + (-1)(1 - p) \iff p = \frac{1}{2}$$

- (See figure after next slide)

# Matching pennies

- Player 2
  - 1 We know that if  $p > \frac{1}{2}$ , player 1 is likely playing heads. Then player 2 wants to play tails instead, i.e.,  $q = 0$ .
  - 2 Go to the figure on the next slide, and draw  $q = 0$  for all  $p > \frac{1}{2}$ .
  - 3 If  $p < \frac{1}{2}$ , player 1 is likely playing tails. Then player 2 wants to play heads, i.e.,  $q = 1$ .
  - 4 Graphically, draw  $q = 1$  for all  $p < \frac{1}{2}$ .

# Matching pennies





# Matching pennies

- We can represent these BRFs as follows:

- **Player 1**

$$BR_1(q) = \begin{cases} \text{Heads if } q > \frac{1}{2} \\ \{\text{Heads, Tails}\} \text{ if } q = \frac{1}{2} \\ \text{Tails if } q < \frac{1}{2} \end{cases}$$

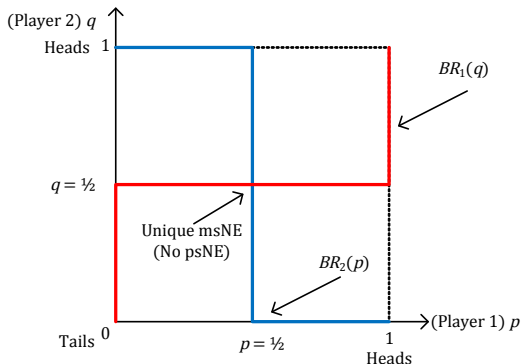
- Player 1 is indifferent between Heads and Tails when  $q$  is exactly  $q = \frac{1}{2}$

- **Player 2**

$$BR_2(p) = \begin{cases} \text{Tails if } p > \frac{1}{2} \\ \{\text{Heads, Tails}\} \text{ if } p = \frac{1}{2} \\ \text{Heads if } p < \frac{1}{2} \end{cases}$$

- Player 2 is indifferent between Heads and Tails when  $p$  is exactly  $p = \frac{1}{2}$

# Matching pennies



- **Player 1:** When  $q > \frac{1}{2}$ , Player 1 prefers to play Heads ( $p = 1$ ); otherwise, Tails.
- **Player 2:** When  $p > \frac{1}{2}$ , Player 2 prefers to play Tails ( $q = 0$ ); otherwise, Heads.

## Matching pennies

- Therefore, the msNE of this game can be represented as

$$\left\{ \left( \frac{1}{2}H, \frac{1}{2}T \right), \left( \frac{1}{2}H, \frac{1}{2}T \right) \right\}$$

where the first parenthesis refers to player 1(row player), and the player 2(column player).

## Battle of the sexes

2. **Battle of the sexes** (example of a normal form game with 2 psNE already!):

		<i>Wife</i>	
		$q$	$1 - q$
<i>Husband</i>	$p$ Football	<u>3</u> , <u>1</u>	0, 0
	$1 - p$ Opera	0, 0	<u>1</u> , <u>3</u>

If the Husband is using a mixed strategy, it must be that he is indifferent between Football and Opera:

$$\begin{aligned}EU_1(F) &= EU_1(O) \\3q + 0(1 - q) &= 0q + 1(1 - q) \\3q &= 1 - q \\4q &= 1 \implies q = \frac{1}{4}\end{aligned}$$

## Battle of the sexes

Similarly, if the Wife is using a mixed strategy, it must be that she is indifferent between Football and Opera:

$$EU_2(F) = EU_2(O)$$

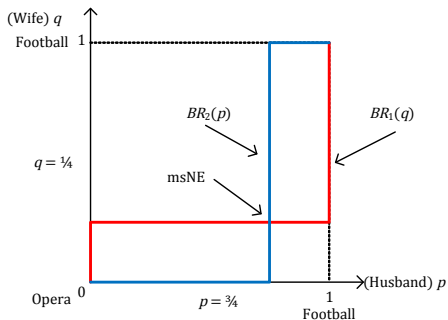
} Practice!

$$p = \frac{3}{4}$$

Therefore, the msNE of this game can be represented as

$$\text{msNE} = \left\{ \underbrace{\left( \frac{3}{4}F, \frac{1}{4}O \right)}_{\text{Husband}}, \underbrace{\left( \frac{1}{4}F, \frac{3}{4}O \right)}_{\text{Wife}} \right\}$$

# Battle of the sexes



- **Husband:** When  $q > \frac{1}{4}$ , he prefers to go to the Football game ( $p = 1$ ); otherwise, the Opera.
- **Wife:** When  $p > \frac{3}{4}$ , she prefers to go to the Football game ( $q = 1$ ); otherwise, the Opera.

# Battle of the sexes

- Best Responses for Battle of the Sexes are hence:
  - **Player 1 (Husband)**

$$BR_1(q) = \begin{cases} \text{Football if } q > \frac{1}{4} \\ \{\text{Football, Opera}\} \text{ if } q = \frac{1}{4} \\ \text{Opera if } q < \frac{1}{4} \end{cases}$$

- **Player 2 (Wife)**

$$BR_2(p) = \begin{cases} \text{Football if } p > \frac{3}{4} \\ \{\text{Football, Opera}\} \text{ if } p = \frac{3}{4} \\ \text{Opera if } p < \frac{3}{4} \end{cases}$$

# Battle of the sexes

- Note the differences in the cutoffs: They reveal each player's preferences.
  - **Husband:** "I will go to the football game as long as there is a slim probability that my wife will be there."
  - **Wife:** "I will only go to the football game if there is more than a 75% chance my husband will be there."



## Prisoner's Dilemma

### 3. Prisoner's Dilemma (One psNE, but are there any msNE?):

		<i>Player 2</i>	
		<i>q</i>	<i>1 - q</i>
<i>Player 1</i>		Confess	Not Confess
		<i>p</i> Confess	<u>-5, -5</u> <u>0, -15</u>
		<i>1 - p</i> Not Confess	-15, <u>0</u> -1, -1

If the first player is using a mixed strategy, it must be that he indifferent between Confess and Not Confess:

$$\begin{aligned}EU_1(C) &= EU_1(NC) \\-5q + 0(1 - q) &= -15q + (-1)(1 - q) \\-5q &= -15q - 1 + q \\9q &= -1 \implies q = -\frac{1}{9}?\end{aligned}$$

## Prisoner's Dilemma

- Similarly, if player 2 is using a mixed strategy, it must be that she is indifferent between Confess and Not Confess:

$$\begin{aligned}EU_2(C) &= EU_2(NC) \\ -5p + 0(1-p) &= -15p + (-1)(1-p) \\ -5p &= -15p - 1 + p \\ 9p &= -1 \implies p = -\frac{1}{9}\end{aligned}$$

- Hence, such msNE would not assign any positive weight to strategies that are strictly dominated.
  - Some textbooks refer to this result by saying that "the support of the msNE is positive only for strategies that are not strictly dominated."

# Tennis game (msNE with three available strategies)

4. **Tennis game** (No psNE, but how do we operate with 3 strategies?):

		<i>Player 2</i>			
		<i>q</i>	<i>1 - q</i>		
		F	C	B	
<i>Player 1</i>	<i>p</i>	F	0, <u>5</u>	2, 3	2, 3
		C	2, 3	1, <u>5</u>	<u>3</u> , 2
	<i>1 - p</i>	B	<u>5</u> , 0	<u>3</u> , 2	2, <u>3</u>

- Remember this game? We used it as an example of how to delete an strategy that was strictly dominated by the combination of two strategies of that player.
  - Let's do it again.

# Tennis game (msNE with three available strategies)

- F is strictly dominated for Player 1:

		Player 2		
		F	C	B
Player 1	F	0, 5	2, 3	2, 3
	$\frac{1}{3}C, \frac{2}{3}B$	4, 1	$\frac{7}{3}, 3$	$\frac{7}{3}, \frac{8}{3}$

$\frac{1}{3}(2) + \frac{2}{3}(5) = \frac{12}{3} = 4$

$\frac{1}{3}(3) + \frac{2}{3}(0) = 1$

$\frac{1}{3}(1) + \frac{2}{3}(3) = \frac{7}{3}$

$\frac{1}{3}(5) + \frac{2}{3}(2) = \frac{9}{3} = 3$

$\frac{1}{3}(3) + \frac{2}{3}(2) = \frac{7}{3}$

$\frac{1}{3}(2) + \frac{2}{3}(3) = \frac{8}{3}$

- We can hence rule out  $F$  from Player 1 because it is strictly dominated by  $(\frac{1}{3}C, \frac{2}{3}B)$ .

## Tennis game (msNE with three available strategies)

- After deleting  $F$  from Player 1's available actions, we are left with:

		<i>Player 2</i>		
		F	C	B
<i>Player 1</i>	C	2, 3	1, 5	3, 2
	B	5, 0	3, 2	2, 3

- Where we can rule out  $F$  from Player 2 because of being strictly dominated by  $C$ .

# Tennis game (msNE with three available strategies)

- Once strategy F has been deleted for both players, we are left with:

		<i>Player 2</i>	
		<i>q</i>	<i>1 - q</i>
<i>Player 1</i>	<i>p</i> C	1, <u>5</u>	<u>3</u> , 2
	<i>1 - p</i> B	<u>3</u> , 2	2, <u>3</u>

- But we cannot identify any psNE, Let's check for msNE:
- If the first player is using a mixed strategy, it must be that he indifferent between C and B:

$$EU_1(C) = EU_1(B) \quad \dots$$

} Practice!

$$q = \frac{1}{3}$$

## Tennis game (msNE with three available strategies)

- Similarly, if player 2 is using a mixed strategy, it must be that she is indifferent between C and B:

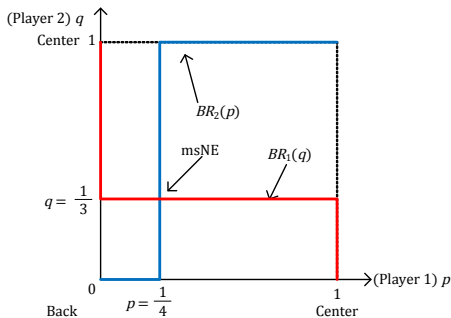
$$EU_2(C) = EU_2(NC) \dots$$

} Practice!

$$p = \frac{1}{4}$$

- (See figure on next slide)

# Tennis game (msNE with three available strategies)



- **Player 1:** If  $q > \frac{1}{3}$ , then Player 1 prefers Back ( $p = 0$ ); otherwise Center.
- **Player 2:** If  $p > \frac{1}{4}$ , then Player 2 prefers Center ( $q = 1$ ); otherwise Back.



# Tennis game (msNE with three available strategies)

- Best Responses in the Tennis Game

- **Player 1**

$$BR_1(q) = \begin{cases} \text{Back if } q > \frac{1}{4} \\ \{\text{Center, Back}\} \text{ if } q = \frac{1}{4} \\ \text{Center if } q < \frac{1}{4} \end{cases}$$

- (Recall that  $p = 0$  implies playing strategy back with probability one).

- **Player 2**

$$BR_2(p) = \begin{cases} \text{Center if } p > \frac{1}{4} \\ \{\text{Center, Back}\} \text{ if } p = \frac{1}{4} \\ \text{Back if } p < \frac{1}{4} \end{cases}$$

# Graphical representation of BRFs and msNE:

- 1 Matching pennies (Done ✓)
- 2 Battle of the sexes (coordination) (Done ✓)
- 3 Additional practice:
  - 1 Lobbying game (Watson page 124).
  - 2 Chicken game (anticoordination).

## A few tricks we just learned...

- **Indifference:** If it is optimal to randomize over a collection of pure strategies, then a player receives the same expected payoff from each of those pure strategies.
  - He must be indifferent between those pure strategies over which he randomizes.
- **Odd number:** In almost all finite games (games with a finite set of players and available actions), there is a finite and odd number of equilibria.
  - *Examples:* 1 NE in matching pennies (only one msNE), 3 NE in BoS (two psNE, one msNE), 1 in PD (only one psNE), etc.
- **Never use strictly dominated strategies:** If a pure strategy does not survive the IDSDS, then a NE assigns a zero probability to that pure strategy.
  - *Example:* PD game, where NC is strictly dominated, it does not receive any positive probability.

## What if players have three undominated strategies?

- Consider the rock-paper-scissors game

		<i>Player 2</i>		
		Rock	Paper	Scissors
<i>Player 1</i>	Rock	0, 0	-1, <u>1</u>	<u>1</u> , -1
	Paper	<u>1</u> , -1	0, 0	-1, <u>1</u>
	Scissors	-1, <u>1</u>	<u>1</u> , -1	0, 0

- First, note that neither player selects a pure strategy (with 100% probability).

## What if players have three undominated strategies?

- Second, every player must be mixing between all his three possible actions, R, P and S.

If Player 1 only mixes between Rock and Paper

		<i>Player 2</i>		
		Rock	Paper	Scissors
<i>Player 1</i>	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

- **Otherwise:** if P1 mixes only between Rock and Paper, then Player 2 prefers to respond with Paper rather than Rock.
- But if Player 2 never uses Rock, then Player 1 gets a higher payoff with Scissors than Paper. **Contradiction!**
- Then players cannot be mixing between only two of their available strategies.

## What if players have three undominated strategies?

- Are you suspecting that the msNE is  $\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ? You're right!

		<i>Player 2</i>		
		Rock	Paper	Scissors
<i>Player 1</i>	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

- We must make every player indifferent between using Rock, Paper, or Scissors.
- That is,  $u_1(\text{Rock}, \sigma_2) = u_1(\text{Paper}, \sigma_2) = u_1(\text{Scissors}, \sigma_2)$  for Player 1, and
- $u_2(\sigma_1, \text{Rock}) = u_2(\sigma_1, \text{Paper}) = u_2(\sigma_1, \text{Scissors})$  for Player 2.

## What if players have three undominated strategies?

- Let's separately find each of these expected utilities.
- If player 1 chooses Rock (first row), he obtains

$$\begin{aligned}u_1(\text{Rock}, \sigma_2) &= 0\sigma_2(R) + (-1)\sigma_2(P) + 1(1 - \sigma_2(R) - \sigma_2(P)) \\ &= -1\sigma_2(P) + 1 - \sigma_2(R) - \sigma_2(P)\end{aligned}$$

		<i>Player 2</i>		
		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	$1 - \sigma_2(R) - \sigma_2(P)$ Scissors
<i>Player 1</i>	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

## What if players have three undominated strategies?

- If player 1 chooses Paper (second row), he obtains

$$\begin{aligned}u_1(\text{Paper}, \sigma_2) &= 1\sigma_2(R) + 0\sigma_2(P) + (-1)(1 - \sigma_2(R) - \sigma_2(P)) \\ &= \sigma_2(R) - 1 + \sigma_2(R) + \sigma_2(P)\end{aligned}$$

		<i>Player 2</i>		
		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	$1 - \sigma_2(R) - \sigma_2(P)$ Scissors
<i>Player 1</i>	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

Second Row

*Player 1*



## What if players have three undominated strategies?

- If player 1 chooses Scissors (third row), he obtains

$$\begin{aligned}u_1(\text{Scissors}, \sigma_2) &= (-1)\sigma_2(R) + 1\sigma_2(P) + 0(1 - \sigma_2(R) - \sigma_2(P)) \\ &= -\sigma_2(R) + \sigma_2(P)\end{aligned}$$

		<i>Player 2</i>		
		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	$1 - \sigma_2(R) - \sigma_2(P)$ Scissors
<i>Player 1</i>	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

Third Row

## What if players have three undominated strategies?

- Making the three expected utilities

$$u_1(\text{Rock}, \sigma_2) = -1\sigma_2(P) + 1 - \sigma_2(R) - \sigma_2(P),$$

$$u_1(\text{Paper}, \sigma_2) = \sigma_2(R) - 1 + \sigma_2(R) + \sigma_2(P), \text{ and}$$

$$u_1(\text{Scissors}, \sigma_2) = -\sigma_2(R) + \sigma_2(P)$$

equal to each other, we obtain

$$\sigma_2(R) = \sigma_2(P) = 1 - \sigma_2(R) - \sigma_2(P)$$

- Hence, player 2 assigns the same probability weights to his three available actions, thus implying

$$\sigma_2^* = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

- A similar argument is applicable to player 1, since players' payoffs are symmetric.