Rationalizability and Nash Equilibrium

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Week 3

Best Response

- Given the previous three problems when applying dominated strategies, let's examine another solution concept:
    - Using Best responses to find Rationalizable strategies, and Nash equilibria.

Best Response

- **Best response:**
  - A strategy s_i^* is a best response of player i to a strategy profile s_{-i} selected by all other players if it provides player i a larger payoff than any of his available strategies $s_i \in S_i$.

    $$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_i \in S_i$$

  - For two players, s_1^* is a best response to a strategy s_2 selected by player 2 if

    $$u_1(s_1^*, s_2) \geq u_1(s_1, s_2) \text{ for all } s_1 \in S_1$$

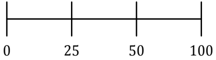
    That is, when player 2 selects s_2, the utility player 1 obtains from playing s_1^* is higher than by playing any other of his available strategies.

Rationalizable strategies

- Given the definition of a best response for player i, we can interpret that he will never use a strategy that cannot be rationalized for any beliefs about his opponents' strategies:
  - A strategy $s_i \in S_i$ is **never a best response** for player i if there are no beliefs he can sustain about the strategies that his opponents will select, s_{-i}, for which s_i is a best response.
  - We can then eliminate strategies that are never a best response from S_i, as they are not rationalizable.
- In fact, the only strategies that are rationalizable are those that survive such iterative deletion, as we define next:
  - A strategy profile $(s_1^*, s_2^*, ..., s_N^*)$ is **rationalizable** if it survives the iterative elimination of those strategies that are never a best response.
- Examples, and comparison with IDSDS (see Handout).

Rationalizable Strategies - Example

1. Beauty Contest / Guess the Average $[0, 100]$



The guess which is closest to $\frac{1}{2}$ the average wins a prize.

"Level 0"Players$\longrightarrow$They select a random number from $[0, 100]$, implying an average of 50.

$$\begin{array}{lcl} \text{"Level 1" Players} & \longrightarrow & BR(s_{-i}) = BR(50) = 25 \\ \text{"Level 2" Players} & \longrightarrow & BR(s_{-1}) = BR(25) = 12.5 \\ \ldots & \longrightarrow & 0 \end{array}$$

Rationalizable Strategies

How many degrees of iteration do subjects use in experimental settings?

- About 1-2 for "regular" people.
    - So they say $s_i = 50$ or $s_i = 25$.
- But...
    - One step more for undergrads who took game theory;
    - One step more for Portfolio managers;
    - 1-2 steps more for Caltech Econ majors;
    - About 3 more for usual readers of financial newspapers (*Expansión* in Spain and *FT* in the UK).

For more details, see Rosemarie Nagel "Unraveling in Guessing Games: An Experimental Study" (1995). *American Economic Review*, pp. 1313-26.

Nash equilibrium

- Besides rationalizability, we can use best responses to identify the Nash equilibria of a game, as we do next.

Nash equilibrium

- A strategy profile $(s_1^*, s_2^*, ..., s_N^*)$ is a Nash equilibrium if every player's strategy is a best response to his opponent's strategies, i.e., if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for all } s_i \in S_i \text{ and for every player } i$$

- For two players, a strategy pair (s_1^*, s_2^*) is a Nash equilibrium if
  - Player 1's strategy, s_1^*, is a best response to player 2's strategy s_2^*,

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \text{ for all } s_1 \in S_1 \implies BR_1(s_2^*) = s_1^*$$

  - and similarly, player 2's strategy, s_2^*, is a best response to player 1's strategy s_1^*,

$$u_2(s_1^*, s_2^*) \geq u_2(s_1, s_2^*) for all s_2 \in S_2 \implies BR_2(s_1^*) = s_2^*$$

Nash equilibrium

- In short, every player must be playing a best response against his opponent's strategies, and
- Players' conjectures must be correct in equilibrium
    - Otherwise, players would have incentives to modify their strategy.
    - This didn't need to be true in the definition of Rationalizability, where beliefs could be incorrect.
- The Nash equilibrium strategies are stable, since players don't have incentives to deviate.

Nash equilibrium

- *Note:*
    - While we have described the concept of best response and Nash equilibrium for the case of pure strategies (no randomizations), our definitions and examples can be extended to mixed strategies too.
    - We will next go over several examples of pure strategy Nash equilibria (psNE) and afterwards examine mixed strategy Nash equilibria (msNE).

Example 1: Prisoner's Dilemma

If Player 2 confesses,
$BR_1(C)=C$

		Player 2	
		Confess	Not Confess
Player 1	Confess	$\underline{-5}$,-5	0,-15
	Not Confess	-15,0	-1,-1

- Let's start analyzing player 1's best responses.
- If player 2 selects Confess (left column), then player 1's best response is to confess as well.
- For compactness, we represent this result as $BR_1(C) = C$, and underline the payoff that player 1 would obtain after selecting his best response in this setting, i.e., -5.

Example 1: Prisoner's Dilemma

If Player 2 does not confess,
$BR_1(NC)=C$

Player 2

		Confess	Not Confess
Player 1	Confess	-5,-5	$\underline{0}$,-15
	Not Confess	-15,0	-1,-1

- Let's continue analyzing player 1's best responses.
- If player 2 selects, instead, Not Confess (right column), then player 1's best response is to confess.
- For compactness, we represent this result as $BR_1(NC) = C$, and underline the payoff that player 1 would obtain after selecting his best response in this setting, i.e., 0.

Example 1: Prisoner's Dilemma

If Player 1 confesses,
$BR_2(C)=C$

		Player 2	
		Confess	Not Confess
Player 1	Confess	-5,$\underline{-5}$	0,-15
	Not Confess	-15,0	-1,-1

- Let's now move to player 2's best responses.
- If player 1 selects Confess (upper row), then player 2's best response is to confess.
- For compactness, we represent $BR_2(C) = C$, and underline the payoff that player 2 would obtain after selecting his best response in this setting, i.e., -5.

Example 1: Prisoner's Dilemma

If Player 1 does not confess,
$BR_2(NC) = C$

		Player 2	
		Confess	Not Confess
Player 1	Confess	-5,-5	0,-15
	Not Confess	-15,$\underline{0}$	-1,-1

- Finally, if player 1 selects Not Confess (lower row), then player 2's best response is to confess.
- For compactness, we represent $BR_2(NC) = C$, and underline the payoff that player 2 would obtain after selecting his best response in this setting, i.e., 0.

Example 1: Prisoner's Dilemma

- Underlined payoffs hence represent the payoffs that players obtain when playing their best responses.
- When we put all underlined payoffs together in the prisoner's dilemma game...

		Player 2	
		Confess	Not Confess
Player 1	Confess	$\underline{-5}$,$\underline{-5}$	$\underline{0}$,-15
	Not Confess	-15,$\underline{0}$	-1,-1

- We see that there is only one cell where the payoffs of both player 1 and 2 were underlined.
- In this cell, players must be selecting mutual best responses, implying that this cell is a Nash equilibrium of the game.
- Hence, we say that the NE of this game is (Confess, Confess) with a corresponding equilibrium payoff of $(-5, -5)$.

Example 2: Battle of the Sexes

- Recall that this is an example of a coordination game, such as those describing technology adoption by two firms.

		Wife	
		Football	Opera
Husband	Football	$\underline{3}, 1$	$0, 0$
	Opera	$0, 0$	$\underline{1}, 3$

- **Husband's best responses:**
  - When his wife selects the Football game, his best response is to also go to the Football game, i.e., $BR_H(F) = F$.
  - When his wife selects Opera, his best response is to also go to the Opera, i.e., $BR_H(O) = O$.

Example 2: Battle of the Sexes

		Wife	
		Football	Opera
Husband	Football	$3, \underline{1}$	$0, 0$
	Opera	$0, 0$	$1, \underline{3}$

- **Wife's best responses:**
  - When her husband selects the Football game, her best response is to also go to the Football game, i.e., $BR_W(F) = F$.
  - When her husband selects Opera, her best response is to also go to the Opera, i.e., $BR_W(O) = O$.

Example 2: Battle of the Sexes

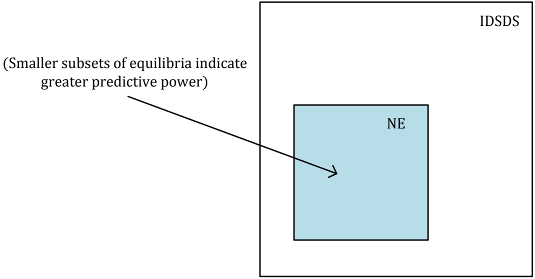
		Wife	
		Football	Opera
Husband	Football	$\underline{3}, \underline{1}$	$0, 0$
	Opera	$0, 0$	$\underline{1}, \underline{3}$

- Two cells have all payoffs underlined. These are the two Nash equilibria of this game:
  - (Football, Football) with equilibrium payoff $(3, 1)$, and
  - (Opera, Opera) with equilibrium payoff $(1, 3)$.

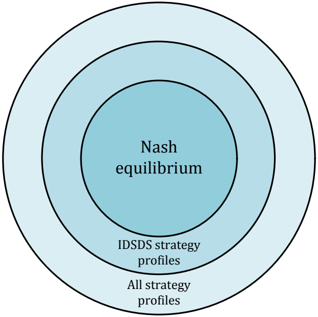
- Prisoner's Dilemma $\longrightarrow$ NE $=$ set of strategies surviving IDSDS
- Battle of the Sexes $\longrightarrow$ NE is a subset of strategies surviving IDSDS (the entire game).

Therefore, NE has more predictive power than IDSDS.

- Great!



The NE provides more precise equilibrium predictions:



Hence, if a strategy profile (s_1^*, s_2^*) is a NE, it must survive IDSDS. However, if a strategy profile (s_1^*, s_2^*) survives IDSDS, it does not need to be a NE.

Example 3: Pareto coordination

		Player 2	
		Tech A	Tech B
Player 1	Tech A	$\underline{2}, \underline{2}$	0, 0
	Tech B	0, 0	$\underline{1}, \underline{1}$

- While we can find two NE in this game,(A,A) and (B,B), there are four strategy profiles surviving IDSDS
  - Indeed, since no player has strictly dominated strategies, all columns and rows survive the application of IDSDS.

Example 3: Pareto coordination

		Player 2	
		Tech A	Tech B
Player 1	Tech A	$\underline{2}, \underline{2}$	0, 0
	Tech B	0, 0	$\underline{1}, \underline{1}$

- While two NE can be sustained, (B,B) yields a lower payoff than (A,A) for both players.
- Equilibrium (B,B) occurs because, once a player chooses B, his opponent is better off at B than at A.
- In other words, they would have to sumultaneously move to A in order to increase their payoffs.

Example 3: Pareto coordination

- Such a miscoordination into the "bad equilibrium" (B,B) is more recurrent than we think:
  - Betamax vs. VHS (where VHS plays the role of the inferior technology B, and Betamax that of the superior technology A). Indeed, once all your friends have VHS, your best response is to buy a VHS as well.
  - Mac vs. PC (before files were mostly compatible).
  - Blu-ray vs. HD-DVD.

Example 4: Anticoordination Game

- The game of chicken is an example of an anticoordination game.

		Dean	
		Swerve	Straight
James	Swerve	0, 0	-1, 1
	Straight	1, -1	-2, -2

- **James' best responses:**
    - When Dean selects Swerve, James' best response is to drive Straight, i.e., $BR_J(\mathit{Swerve}) = \mathit{Straight}$.
    - When Dean selects Straight, James' best response is to Swerve, i.e., $BR_J(\mathit{Straight}) = \mathit{Swerve}$.

Example 4: Anticoordination Game

		Dean	
		Swerve	Straight
James	Swerve	0, 0	-1, $\underline{1}$
	Straight	1, $\underline{-1}$	-2, -2

- **Dean's best responses:**
  - When James selects Swerve, Dean's best response is to drive Straight, i.e., $BR_D(Swerve) = Straight$.
  - When James selects Straight, Dean's best response is to Swerve, i.e., $BR_D(Straight) = Swerve$.

Example 4: Anticoordination Game

		Dean	
		Swerve	Straight
James	Swerve	0, 0	$\underline{-1}$, $\underline{1}$
	Straight	$\underline{1}$, $\underline{-1}$	-2, -2

- Two cells have all payoffs underlined. These are the two NE of this game:
  - (Swerve, Straight) with equilibrium payoff (-1,1), and
  - (Straight, Swerve) with equilibrium payoff (1,-1).
- Unline in coordination games, such as the Battle of the Sexes or technology games, here every player seeks to choose the opposite strategy of his opponent.

Some Questions about NE:

1. Existence? $\longrightarrow$ all the games analyzed in this course will have at least one NE (in pure or **mixed** strategies)
2. Uniqueness? $\longrightarrow$ Small predictive power. Later on we will learn how to restrict the set of NE.

Example 6: Rock-Paper-Scissors

- Not all games must have one NE using pure strategies...

			Lisa	
		Rock	Paper	Scissors
	Rock	0, 0	-1, 1	$\underline{1}$, -1
Bart	Paper	$\underline{1}$, -1	0, 0	-1, 1
	Scissors	-1, 1	$\underline{1}$, -1	0, 0

- **Bart's best responses:**
  - If Lisa chooses Rock, then Bart's best response is to choose Paper, i.e., $BR_B(Rock) = Paper$.
  - If Lisa chooses Paper, then Bart's best response is to choose Scissors, i.e., $BR_B(Paper) = Scissors$.
  - If Lisa chooses Scissors, then Bart's best response is to choose Rock, i.e., $BR_B(Scissors) = Rock$.

Example 6: Rock-Paper-Scissors

		Lisa		
		Rock	Paper	Scissors
	Rock	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
Bart	Paper	$\underline{1}$, -1	0, 0	-1, $\underline{1}$
	Scissors	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- **Lisa's best responses:**
  - If Bart chooses Rock, then Lisa's best response is to choose Paper, i.e., $BR_L(Rock) = Paper$.
  - If Bart chooses Paper, then Lisa's best response is to choose Scissors, i.e., $BR_L(Paper) = Scissors$.
  - If Bart chooses Scissors, then Lisa's best response is to choose Rock, i.e., $BR_L(Scissors) = Rock$.

Example 6: Rock-Paper-Scissors

			Lisa	
		Rock	Paper	Scissors
	Rock	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
Bart	Paper	$\underline{1}$, -1	0, 0	-1, $\underline{1}$
	Scissors	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- In this game, there are no NE using pure strategies!
  - But it will have a NE using mixed strategies (In a couple of weeks).

Example 7: Game with Many Strategies

		Player 2			
		w	x	y	z
Player 1	a	0, 1	0, 1	1, 0	$\underline{3}$, 2
	b	1, 2	$\underline{2}$, 2	$\underline{4}$, 0	0, 2
	c	2, 1	0, 1	1, 2	1, 0
	d	$\underline{3}$, 0	1, 0	1, 1	$\underline{3}$,1

- **Player 1's best responses:**
    - If Player 2 chooses w, then Player 1's best response is to choose d, i.e., $BR_1(w) = d$.
    - If Player 2 chooses x, then Player 1's best response is to choose b, i.e., $BR_1(x) = b$.
    - If Player 2 chooses y, then Player 1's best response is to choose b, i.e., $BR_1(y) = b$.
    - If Player 2 chooses z, then Player 1's best response is to choose a or d, i.e., $BR_1(z) = \{a, d\}$.

Example 7: Game with Many Strategies

		Player 2			
		w	x	y	z
Player 1	a	0, 1	0, 1	1, 0	3, $\underline{2}$
	b	1, $\underline{2}$	2, $\underline{2}$	4, 0	0, $\underline{2}$
	c	2, 1	0, 1	1, $\underline{2}$	1, 0
	d	3, 0	1, 0	1, $\underline{1}$	3,$\underline{1}$

- **Player 2's best responses:**
  - If Player 1 chooses a, then Player 2's best response is to choose z, i.e., $BR_1(a) = z$.
  - If Player 1 chooses b, then Player 2's best response is to choose w, x or z, i.e., $BR_1(b) = \{w, x, z\}$.
  - If Player 1 chooses c, then Player 2's best response is to choose y, i.e., $BR_1(c) = y$.
  - If Player 1 chooses d, then Player 2's best response is to choose y or z, i.e., $BR_1(d) = \{y, z\}$.

Example 7: Game with Many Strategies

		Player 2			
		w	x	y	z
Player 1	a	0, 1	0, 1	1, 0	$\underline{3}$, $\underline{2}$
	b	1, $\underline{2}$	$\underline{2}$, $\underline{2}$	$\underline{4}$, 0	0, $\underline{2}$
	c	2, 1	0, 1	1, $\underline{2}$	1, 0
	d	$\underline{3}$, 0	1, 0	1, $\underline{1}$	$\underline{3}$,$\underline{1}$

- NE can be applied very easily to games with many strategies. In this case, there are 3 seperate NE: (b,x), (a,z) and (d,z).
- Two important points:
  - Note that BR cannot be empty: I might be indifferent among my available strategies, but BR is non-empty.
  - Another important point: Players can use weakly dominated strategies, i.e., a or d by Player 1; y or z by Player 2.

Example 8: The American Idol Fandom

- We can also find the NE in 3-player games.
  - Harrington, pp. 101-102.
  - More generally representing a coordination game between three individuals or firms.
- "Alicia, Kaitlyn, and Lauren are ecstatic. They've just landed tickets to attend this week's segment of American Idol. The three teens have the same favorite among the nine contestants that remain: Ace Young. They're determined to take this opportunity to make a statement. While [text]ing, they come up with a plan to wear T-shirts that spell out "ACE" in large letters. Lauren is to wear a T-shirt with a big "A," Kaitlyn with a "C," and Alicia with an "E." If they pull this stunt off, who knows–they might end up on national television! OMG!

Example 8: The American Idol Fandom

- While they all like this idea, each is tempted to wear instead an attractive new top just purchased from their latest shopping expedition to Bebe. It's now an hour before they have to leave to meet at the studio, and each is at home trying to decide between the Bebe top and the lettered T-shirt. What should each wear?"

Alicia chooses E

		Kaitlyn	
		C	Bebe
Lauren	A	2, 2, 2	0, 1, 0
	Bebe	1, 0, 0	1, 1, 0

Alicia chooses Bebe

		Kaitlyn	
		C	Bebe
Lauren	A	0, 0, 1	0, 1, 1
	Bebe	1, 0, 1	1, 1, 1

Example 8: The American Idol Fandom

Alicia chooses E

Kaitlyn

		C	Bebe
Lauren	A	$\underline{2}, \underline{2}, \underline{2}$	$0, 1, 0$
	Bebe	$1, 0, 0$	$\underline{1}, \underline{1}, 0$

Alicia chooses Bebe

Kaitlyn

		C	Bebe
Lauren	A	$0, 0, 1$	$0, \underline{1}, \underline{1}$
	Bebe	$\underline{1}, 0, \underline{1}$	$\underline{1}, \underline{1}, \underline{1}$

- There are 2 psNE: (A,C,E) and (Bebe, Bebe, Bebe)

Example 9: Voting: Sincere or Devious?

- Harrington pp. 102-106
- Three shareholders (1, 2, 3) must vote for three options (A, B, C) where
  - Shareholder 1 controls 25% of the shares
  - Shareholder 2 controls 35% of the shares
  - Shareholder 3 controls 40% of the shares
- Their preferences are as follows:

Shareholder	1st Choice	2nd Choice	3rd Choice
1	A	B	C
2	B	C	A
3	C	B	A

Example 9: Voting: Sincere or Devious?

3 votes for A

2

		A	B	C
1	A	A	A	A
	B	A	B	A
	C	A	A	C

3 votes for B

2

		A	B	C
1	A	A	B	B
	B	B	B	B
	C	B	B	C

3 votes for C

2

		A	B	C
1	A	A	C	C
	B	C	B	C
	C	C	C	C

This implies the following winners, for each possible strategy profile:

Example:
1 votes B, 2 votes B, 3 votes C:
Votes for B = 25 + 35 = 60%
Votes for C = 40%

B is the Winner

Example 9: Voting: Sincere or Devious?

3 votes for A

		2		
		A	B	C
1	A	2,0,0	2,0,0	2,0,0
	B	2,0,0	1,2,1	2,0,0
	C	2,0,0	2,0,0	0,1,2

3 votes for B

		2		
		A	B	C
1	A	2,0,0	1,2,1	1,2,1
	B	1,2,1	1,2,1	1,2,1
	C	1,2,1	1,2,1	0,1,2

3 votes for C

		2		
		A	B	C
1	A	2,0,0	0,1,2	0,1,2
	B	0,1,2	1,2,1	0,1,2
	C	0,1,2	0,1,2	0,1,2

Each player obtains a payoff of:
2 if his most preferred option is adopted
1 if his second most preferred option is adopted
0 if his least preferred option is adopted

Example 9: Voting: Sincere or Devious?

3 votes for A

2

		A	B	C
1	A	$\underline{2},\underline{0},\underline{0}$	$\underline{2},\underline{0},0$	$\underline{2},\underline{0},0$
	B	$\underline{2},0,0$	$1,\underline{2},\underline{1}$	$\underline{2},0,0$
	C	$\underline{2},0,0$	$\underline{2},0,0$	$0,\underline{1},\underline{2}$

3 votes for B

2

		A	B	C
1	A	$\underline{2},0,\underline{0}$	$\underline{1},\underline{2},1$	$\underline{1},\underline{2},1$
	B	$1,\underline{2},1$	$\underline{1},\underline{2},\underline{1}$	$\underline{1},\underline{2},1$
	C	$1,\underline{2},1$	$\underline{1},\underline{2},1$	$0,1,\underline{2}$

3 votes for C

2

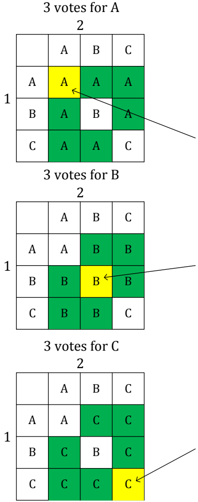
		A	B	C
1	A	$\underline{2},0,\underline{0}$	$0,\underline{1},\underline{2}$	$\underline{0},\underline{1},\underline{2}$
	B	$0,1,\underline{2}$	$\underline{1},\underline{2},\underline{1}$	$\underline{0},1,\underline{2}$
	C	$0,\underline{1},\underline{2}$	$0,\underline{1},\underline{2}$	$\underline{0},\underline{1},\underline{2}$

5 NEs:
(A, A, A)
(B, B, B)
(C, C, C)
(B, B, C)
(A, C, C)

A comment on the NEs we just found

- **First point** :Sincere voting cannot be supported as a NE of the game.
    - Indeed, for sincere voting to occur, we need that each player selects his/her most preferred option, i.e., profile (A,B,C), which is not a NE.
- **Second point:** In the symmetric strategy profiles (A,A,A), (B,B,B), and (C,C,C), no player is pivotal, since the outcome of the election does not change if he/she were to vote for a different option.
    - That is, a player's equilibrium action is weakly dominant.

A comment on the NEs we just found



3 votes for A

2

		A	B	C
	A	A	A	A
1	B	A	B	A
	C	A	A	C

3 votes for B

2

		A	B	C
	A	A	B	B
1	B	B	B	B
	C	B	B	C

3 votes for C

2

		A	B	C
	A	A	C	C
1	B	C	B	C
	C	C	C	C

This result can be easily visualized by analyzing the matrices representing the results of each voting profile.

For instance, for (A,A,A), option A arises. If only one player changes his vote, option A is still the winner.

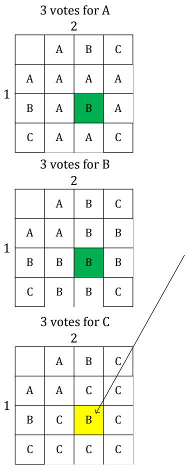
A similar argument is applicable to (B,B,B) ...

... and to (C,C,C).

A comment on the NEs we just found

- **Third point:** Similarly, in equilibrium (B,B,C), shareholder 3 does not have incentives to deviate to a vote different than C since he would not be able to change the outcome.
    - Similarly for shareholder 1 in equilibrium (A,C,C).

A comment on the NEs we just found



3 votes for A

2

		A	B	C
	A	A	A	A
1	B	A	🟩 B	A
	C	A	A	C

3 votes for B

2

		A	B	C
	A	A	B	B
1	B	B	🟩 B	B
	C	B	B	C

3 votes for C

2

		A	B	C
	A	A	C	C
1	B	C	🟨 B	C
	C	C	C	C

In NE (B,B,C), option B is the winner.

In (B,B,C) a unilateral deviation of player 3 towards voting for A (in the top matrix) or for B (in the middle matrix) ***still*** yields option B as the winner. Player 3 therefore has no incentives to unilaterally change his vote.