EconS 501 Final Exam - December 14th, 2020

Show all your work clearly and make sure you justify all your answers.
NAME____________________________

1. Consider a setting in which two firms, $N = 2$, have free access to a fishing ground. Every unit of appropriation (e.g., a ton of fish) is sold in the international market which, for simplicity, is assumed to be perfectly competitive. As a result, every firm takes the market price p as given, which is normalized to $p = \$1$. In this case, consider that firms face a different cost function

$$C_1(q_1, q_2) = \frac{q_1(q_1 + q_2)}{S} \quad \text{and} \quad C_2(q_2, q_1) = \frac{\alpha q_2(q_1 + q_2)}{S}$$

where $\alpha \in [0, 1]$ denotes firm 2's cost advantage. When $\alpha = 1$, both firms face the same cost function. However, when $\alpha < 1$, firm 2 benefits from a cost advantage relative to firm 1. S denotes the stock of the resource. In addition, q_i represents fisherman i's appropriation where $i = \{1, 2\}$.

(a) Find firm 1 best response function, $q_1(q_2)$. How is it affected by an increase in parameter α? Interpret.

- Firm 1's chooses its appropriation level q_1 to solve

$$\max_{q_1 \geq 0} \ \pi_1 = q_1 - \frac{q_1(q_1 + q_2)}{S}.$$

Differentiating with respect to q_1, yields

$$1 - \frac{2q_1 + q_2}{S} = 0$$

and solving for q_1, we find firm 1's best response function

$$q_1(q_2) = \frac{S}{2} - \frac{1}{2}q_2.$$

Intuitively, firm 1 appropriates half of the available stock, $\frac{S}{2}$, when its rivals do not appropriate any units, $q_2 = 0$; but its appropriation decreases as firm 2 appropriates positive amounts, $q_2 > 0$.

(b) Find firm 2 best response function, $q_2(q_1)$. How is it affected by an increase in parameter α? Interpret.

- Firm 2's chooses its appropriation level q_2 to solve

$$\max_{q_2 \geq 0} \ \pi_2 = q_2 - \frac{\alpha q_2(q_1 + q_2)}{S}.$$

Differentiating with respect to q_2, yields

$$1 - \frac{\alpha\,(2q_2 + q_1)}{S} = 0$$

and solving for q_2, we find firm 2's best response function

$$q_2(q_1) = \frac{S}{2\alpha} - \frac{1}{2}q_1.$$

Intuitively, firm 2 appropriates $\frac{S}{2\alpha}$, when its rivals do not appropriate any units, $q_1 = 0$; but its appropriation decreases as firm 1 appropriates positive amounts, $q_1 > 0$. Note that if firm 2 becomes more efficient (decrease in α), it can appropriate more units of the resource.

(c) Compare firm 1 and 2's best response functions.

- Comparing firm 1 and 2's best response function, we find that they have the same slope but $q_1(q_2)$ originates below $q_2(q_1)$ since $\frac{S}{2} \leq \frac{S}{2\alpha}$ given that $2 \geq 2\alpha$ because $\alpha \in [0, 1]$ by assumption. Intuitively, for a given appropriation of its rival, firm 1 appropriates fewer units than firm 2 since the former suffers a cost disadvantage relative to the latter.

(d) Find the equilibrium appropriation pair (q_1^*, q_2^*). [*Hint*: Since firms are asymmetric, you cannot invoke symmetric appropriation as we did in Section 2.3. Instead, you will need to insert the best response function of one of the firms into that of its rival, as when you solve a system of two equations with two unknowns.]

- Inserting firm 2's best response function into firm 1's, we obtain

$$q_1 = \frac{S}{2} - \frac{1}{2}\overbrace{\left(\frac{S}{2\alpha} - \frac{1}{2}q_1\right)}^{q_2(q_1)}.$$

  Rearranging, and solving for q_1, yields firm 1's equilibrium appropriation level

$$q_1^* = \frac{S(2\alpha - 1)}{3\alpha}.$$

- Finally, to find firm 2's equilibrium appropriation level, we can insert q_1^* into firm 2's best response function, $q_2^* = q_2(q_1^*)$, to obtain

$$\begin{aligned} q_2^* &= \frac{S}{2\alpha} - \frac{1}{2}\overbrace{\left(\frac{S(2\alpha - 1)}{3\alpha}\right)}^{q_1^*} \\ &= \frac{S(2-\alpha)}{3\alpha}. \end{aligned}$$

(e) Which firm appropriates more in equilibrium?

- Finding the difference in equilibrium appropriation between firm 2 and 1, yields

$$\begin{aligned} q_2^* - q_1^* &= \frac{S(2-\alpha)}{3\alpha} - \frac{S(2\alpha - 1)}{3\alpha} \\ &= S\frac{1-\alpha}{\alpha} \end{aligned}$$

which is positive since $\alpha \in [0,1]$ by assumption. Therefore, firm 2 appropriates more units than firm 1 given that the former enjoys a cost advantage relative to its rival. In the extreme case in which firm 2 is very efficient, $\alpha = 0$, the difference $q_2^* - q_1^*$ reaches its maximum level. In contrast, if both firms are equally efficient, $\alpha = 1$, the difference $q_2^* - q_1^*$ is nil.

(f) Find the equilibrium appropriation pair when firms are symmetric, $\alpha = 1$ and compare it to your result in part (d).

- When firms are cost symmetric, $\alpha = 1$, their equilibrium appropriations become

$$q_1^* = \frac{S(2-1)}{3} = \frac{S}{3}, \text{ and}$$
$$q_2^* = \frac{S(2-1)}{3} = \frac{S}{3}.$$

In this setting, both firms appropriation is symmetric, $q_1^* = q_2^* = \frac{S}{3}$.

(g) How is equilibrium appropriation affected by an increase in parameter α? Interpret.

- Firm 1's equilibrium appropriation is increasing in α since $\frac{\partial q_1^*}{\partial \alpha} = \frac{5}{3\alpha^2} > 0$. Intuitively, as firm 2's cost advantage shrinks (i.e., parameter α approaches one), firm 1 appropriates more units. In contrast, firm 2's equilibrium appropriation is decreasing in α since $\frac{\partial q_2^*}{\partial \alpha} = -\frac{2S}{3\alpha^2} < 0$. Therefore, as firm 2's cost advantage shrinks (i.e., parameter α approaches one), this firm appropriates fewer units.

2. Consider a firm with production technology $f(x)$, where $x \in [0,1]$ stands for the input (e.g, gas supply), and $f(x)$ represents the output that is sold in a competitive market at a price p. The firm is subject to an input cost function $g(x)$ that is increasing and convex in input x. This happens, for example, when there is only one company supplying natural gas.

(a) Define $\varepsilon_g \equiv \frac{\partial x}{\partial g(x)} \frac{g(x)}{x}$ to be the price elasticity of gas supply, measuring the percentage change of gas supply given one percent change in gas price. Setup the firm's profit-maximization problem to maximize $\pi(x) = pf(x) - g(x)x$, and show that

$$pf'(x) = g(x)\left[1 + \frac{1}{\varepsilon_g}\right].$$

- The firm chooses input x to solve the following profit maximization problem,

$$\max_{x \geq 0} \pi(x) = pf(x) - g(x)x$$

Differentiating with respect to x, and assuming interior solutions,

$$pf'(x) - g'(x)x - g(x) = 0$$

Rearranging, we obtain

$$pf'(x) = g(x)\left[1 + \frac{g'(x)\,x}{g(x)}\right]$$

Since $g'(x) = \frac{\partial g(x)}{\partial x}$, the above equality can be written as

$$pf'(x) = g(x)\left[1 + \frac{\partial g(x)}{\partial x}\frac{x}{g(x)}\right]$$

Simplifying, we have

$$pf'(x) = g(x)\left[1 + \frac{1}{\varepsilon_g}\right]$$

(b) Let $f(x) = x$ and $g(x) = x^{\beta}$, where $\beta > 1$. Use the expression found in part (a) to identify the optimal gas supply x^*. For simplicity, you may assume that $p = 1$ in the remainder of this exercise.

- Since $f(x) = x$ and $g(x) = x^{\beta}$, we have $f'(x) = 1$ and $g'(x) = \beta x^{\beta-1}$, yielding a price elasticity of gas supply of

$$\varepsilon_g = \frac{g(x)}{xg'(x)} = \frac{x^{\beta}}{x\beta x^{\beta-1}} = \frac{1}{\beta}$$

Substituting $\varepsilon_g = \frac{1}{\beta}$ into the expression we found in part (a), yields

$$1 = x^{\beta}(1+\beta)$$

Rearranging, the optimal gas supply, x^*, solves

$$x^* = (1+\beta)^{-\frac{1}{\beta}}$$

(c) *Comparative statics.* How does x^* change with β? Explain.

- Differentiating x^* with respect to β, and using the fact that

$$(1+\beta)^{-\frac{1}{\beta}} = \exp\left[-\frac{1}{\beta}\log(1+\beta)\right],$$

we can use Chain rule to derive

$$\frac{\partial x^*}{\partial \beta} = \exp\left[-\frac{1}{\beta}\log(1+\beta)\right]\left[\frac{1}{\beta^2}\log(1+\beta) - \frac{1}{\beta(1+\beta)}\right]$$
$$= (1+\beta)^{-\frac{1}{\beta}}\left[\frac{(1+\beta)\log(1+\beta) - \beta}{\beta^2(1+\beta)}\right]$$

Since $(1+\beta)\log(1+\beta) - \beta > 0$ for all $\beta \geq 1$, we have that $\frac{\partial x^*}{\partial \beta} > 0$, indicating that, as the supply function becomes more convex, gas becomes less costly (remembering that $x \in [0,1]$), so that the firm can use more gas in producing output.

(d) *Numerical example.* Evaluate the firm's optimal gas supply x^* when $\beta = 1$, $\beta = 2$, $\beta = 4$, and $\beta \to +\infty$. Interpret.

- Substituting $\beta = 1$ into x^* yields

$$x^* = (1+1)^{-\frac{1}{1}} = 0.5.$$

- When $\beta = 2$, we obtain that

$$x^* = (1+2)^{-\frac{1}{2}} = 0.58.$$

- When $\beta = 4$, we have that

$$x^* = (1+4)^{-\frac{1}{4}} = 0.67.$$

- When $\beta \to +\infty$, we find that

$$\begin{aligned}
\lim_{\beta\to\infty} x^* &= \lim_{\beta\to\infty} \exp\left[-\frac{\log(1+\beta)}{\beta}\right] \\
&= \exp\left[-\lim_{\beta\to\infty}\frac{\log(1+\beta)}{\beta}\right] \\
&= \exp\left[-\frac{\lim_{\beta\to\infty}\frac{d\log(1+\beta)}{d\beta}}{\lim_{\beta\to\infty}\frac{d\beta}{d\beta}}\right] \\
&= \exp\left[-\lim_{\beta\to\infty}\frac{1}{1+\beta}\right] \\
&= \exp[0] = 1.
\end{aligned}$$

  where we apply the Continuous Mapping Theorem in the second line since the exponential function is continuous in the limit, and the L'Hôpital's rule in the third line since both the numerator and denominator approach positive infinity with β.

- Intuitively, the supply curve, $g(x)$, bends away from the 45^o-line (where $\beta = 1$) when β increases (becoming more convex), ultimately allowing the firm to use more gas as this input becomes less expensive. A similar argument applies with the marginal cost curve, $g'(x)\,x + g(x)$, which becomes more convex as β increases. In the limit, when $\beta \to +\infty$, the marginal cost curve intersects with the marginal revenue product curve, $pf'(x)$, at $x = 1$ unit.

3. Consider an individual with utility function

$$u(x_1, x_2) = \max\{x_1, x_2\}$$

who derives utility from the *maximum* consumption of goods 1 and 2, subject to a budget constraint $p_1x_1 + p_2x_2 \leq w$, where p_1 and p_2 are the prices of goods 1 and 2, respectively, and wealth is $w \geq 0$.

(a) Find the Walrasian demand. (*Hint*: consider two separate cases: (i) $p_1 < p_2$, and (ii) $p_1 \geq p_2$.)

- The individual solves the following UMP:

$$\max_{x_1,x_2} u(x_1,x_2) = \max\{x_1,x_2\}$$

$$\text{subject to } \; p_1x_1 + p_2x_2 \leq w.$$

Since the individual can attain the same utility level by minimizing his consumption of good 1 (good 2) while keeping the consumption of good 2 (good 1) unchanged, he can spend all his wealth in one good to maximize his utility. In comparing consumption bundles $\left(\frac{w}{p_1},0\right)$ and $\left(0,\frac{w}{p_2}\right)$, he consumes only good 1 if and only if it generates a higher utility, $\frac{w}{p_1} > \frac{w}{p_2}$, that simplifies to $p_2 > p_1$. Therefore, he spends all his wealth in good 1 (good 2) if the price of good 1 falls below (is above) the price of good 2, yielding Walrasian demand of

$$x(p,w) = \begin{cases} \frac{w}{p_1} & \text{if } \; p_1 < p_2 \\ \frac{w}{p_2} & \text{if } \; p_1 \geq p_2 \end{cases}$$

(b) Find the value function and show that the Roy's identity holds.

- The value function of the UMP is the indirect utility function

$$v(p,w) = \begin{cases} \max\left\{\frac{w}{p_1},0\right\} = \frac{w}{p_1} & \text{if } \; p_1 < p_2 \\ \max\left\{0,\frac{w}{p_2}\right\} = \frac{w}{p_2} & \text{if } \; p_1 \geq p_2 \end{cases}$$

- Applying the Implicit Function Theorem to the value function, we obtain

$$-\frac{\frac{\partial v(p,w)}{\partial p}}{\frac{\partial v(p,w)}{\partial w}} = \begin{cases} -\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{-\frac{w}{p_1^2}}{\frac{1}{p_1}} \\ -\frac{\frac{\partial v(p,w)}{\partial p_2}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{-\frac{w}{p_2^2}}{\frac{1}{p_2}} \end{cases}$$

which simplifies to

$$-\frac{\frac{\partial v(p,w)}{\partial p}}{\frac{\partial v(p,w)}{\partial w}} = \begin{cases} \frac{w}{p_1} \\ \frac{w}{p_2} \end{cases}$$

which coincides with the Walrasian demand, $x(p,w)$. Therefore, the Roy's Identity holds.

(c) Does the utility function satisfy homotheticity? Interpret your results.

- When the amount of goods 1 and 2 is increased by a common factor μ,

$$u(\mu x_1,\mu x_2) = \max\{\mu x_1,\mu x_2\} = \mu \times \max\{x_1,x_2\} = \mu \times u(x_1,x_2)$$

so that the utility function is homogeneous of degree 1, and thus, homothetic.

- *Intuition*: If the individual increases his consumption of both goods by a common factor μ, his utility increases proportionally (by μ). Specifically, if he was only purchasing units of good 1 (good 2), his purchases of this good are increased by μ, leaving the purchases of the other good unaffected.

- Therefore, when we increase the amount of all goods by a common factor μ, we obtain consumption bundles μx and $\mu x'$, which must also lie on the same indifference curve, so that

$$\begin{aligned} u(\mu x) &= \max\{\mu x_1, \mu x_2\} = \mu \max\{x_1, x_2\} \\ &= \mu \max\{x_1', x_2'\} = \max\{\mu x_1', \mu x_2'\} \\ &= u(\mu x'). \end{aligned}$$

As a result, any two indifference curves have the same slope at the point they are crossed by a ray from the origin.

4. Consider that your preference relation over three bundles, x_1, x_2, and x_3, satisfies

$$\begin{aligned} x_1 &\succ x_2 \\ x_2 &\succ x_3 \\ x_3 &\succ x_1 \end{aligned}$$

(a) Show that you can be wiped out of your wealth w, where $w > 0$. (Hint: Begin with x_3.)

- First, let me begin with $x_2 \succ x_3$. I am willing to pay an amount α, where $\alpha > 0$, to exchange x_3 for x_2, since $x_2 \succ x_3$, ending up with wealth $w - \alpha$. The monetary amount $\alpha > 0$ can be as small as necessary to induce me to exchange x_3 for x_2.
- Second, since $x_1 \succ x_2$, I am willing to pay an amount β, where $\beta > 0$, to exchange x_2 for x_1, ending up with wealth $w - \alpha - \beta$. (The monetary amount β can coincide with α, $\beta = \alpha$, or differ, $\beta \neq \alpha$, without affecting our final result.)
- Third, since $x_3 \succ x_1$, I am willing to pay an amount γ, where $\gamma > 0$, to exchange x_1 for x_3, ending up with wealth $w - \alpha - \beta - \gamma$. (The monetary amount γ can coincide with α, β, or none of them, without affecting our final result.)
- Defining the sum of monetary amounts I paid so far, $y \equiv \alpha + \beta + \gamma$, I end up with my original bundle, x_3, but my wealth is reduced from w to $w - y$. Repeating the above three steps for n rounds, my wealth is wiped out, that is, we can find a number of rounds $\overline{n} \in \mathbb{N}$ such that

$$w - (\overline{n} + 1)\, y < 0 < w - \overline{n} y$$

meaning that, at round $\overline{n}$, my wealth is still positive; but going through another round, $\overline{n} + 1$, will leave me with a negative wealth.

(b) Consider an individual with a preference relation that violates rationality because his preferences are incomplete or intransitive. Discuss.

- A rational preference relation must be complete and transitive. Suppose that my preference relation is, instead:

- incomplete, then there exists a pair of alternatives x and y in set X which I cannot compare, implying that my preference relation is not well defined.
- intransitive, then there exists a "money pump" as in part (a) that can eliminate all my wealth after finite rounds of exchanges.

Thus, preference relation must be complete and transitive in order to be rational.

GOOD LUCK!