

## Recitation #9 - (10/30/2018)

1. Suppose there are two periods, ‘today’ (i.e. period 1) and ‘tomorrow’ (i.e. period 2), and a single consumption good. An individual called Andreu (not necessarily the same person as the first author in Mas-Colell, Whinston and Green’s *Microeconomic Theory*) has preferences over two-period consumption streams that are additively separable. In particular assume his preferences admit a representation of the form:

$$U(x_1, x_2) = u(x_1) + \frac{1}{1+r}u(x_2),$$

where  $r > 0$  is the interest rate at which the agent can borrow or lend money, and

$$u(x_t) = -\frac{(x_t - 2)^2}{2}$$

for every time period  $t = \{1, 2\}$ .

- (a) If his income today is  $y_1 = 1$  and he knows that his income tomorrow will also be  $y_2 = 1$ , and solve for her optimal consumption in each period and calculate the level of discounted lifetime utility he achieves.

- Andreu’s utility maximization problem is to optimally choose  $x_1$  and  $x_2$  to solve

$$\begin{aligned} \max_{x_1, x_2} U(x_1, x_2) &= u(x_1) + \beta u(x_2) \\ \text{subject to } x_1 &= y_1 - s, \text{ and} \\ x_2 &= y_2 + (1+r)s \end{aligned}$$

where  $s$  denotes his savings. Simplifying, we can more compactly express this maximization problem as an unconstrained problem where Andreu only needs to choose a single choice variable,  $s$ , as follows

$$\max_s -\frac{(y_1 - 2 - s)^2}{2} - \frac{1}{1+r} \frac{(y_2 - 2 + (1+r)s)^2}{2}$$

- Taking first order conditions with respect to  $s$ , we obtain

$$(y_1 - 2 - s) - (y_2 - 2 + (1+r)s) = 0$$

Rearranging and solving for  $s$ , we find optimal savings of

$$s = \frac{y_1 - y_2}{2 + r}.$$

- Hence, when  $y_1 = 1$  and  $y_2 = 1$ , Andreu's savings are zero, i.e.,  $s = \frac{1-1}{2+r} = 0$ , and his utility level becomes

$$U(1, 1) = -\frac{1}{2} - \frac{1}{2(1+r)} = -\frac{(2+r)}{2(1+r)}$$

Suppose that Andreu still knows that his income today is  $y_1 = 1$ . He is, however, uncertain about his future payoff  $y_2$ : It could be high,  $y_2^H = 3/2$  or low,  $y_2^L = 1/2$ , each happening with equal probability. His problem now is to maximize his discounted expected utility by choosing the initial period consumption  $x_1$ ; his future consumption if his income tomorrow is high,  $x_2^H$ ; and his future consumption if income tomorrow is low,  $x_2^L$ .

- Formulate the agent's optimization problem and solve for the optimal consumption plan and his level of discounted expected utility.

- Andreu's optimization problem now becomes selecting  $x_1$ ,  $x_2^H$  and  $x_2^L$  to solve

$$\max_{x_1, x_2^H, x_2^L} -\frac{(x_1 - 2)^2}{2} - \frac{1}{1+r} \left[ \frac{1}{2} \frac{(x_2^H - 2)^2}{2} + \frac{1}{2} \frac{(x_2^L - 2)^2}{2} \right]$$

subject to  $x_1 = y_1 - s$ ,

$$x_2^H = y_2^H + (1+r)s, \text{ and}$$

$$x_2^L = y_2^L + (1+r)s.$$

Rearranging, we can express this optimization problem as an unconstrained maximization problem, where now Andreu's single choice variable is his savings,  $s$ , as follows

$$\max_s -\frac{(y_1 - 2 - s)^2}{2} - \frac{1}{1+r} \left[ \frac{1}{2} \frac{(y_2^H - 2 + (1+r)s)^2}{2} + \frac{1}{2} \frac{(y_2^L - 2 + (1+r)s)^2}{2} \right]$$

- Taking first order conditions with respect to  $s$ , we obtain

$$(y_1 - 2 - s) - \frac{1}{2} (y_2^H - 2 + (1+r)s) - \frac{1}{2} (y_2^L - 2 + (1+r)s) = 0$$

rearranging,

$$y_1 - \frac{1}{2}y_2^H - \frac{1}{2}y_2^L = s(2+r),$$

and solving for  $s$  yields

$$s = \frac{y_1 - E[y_2]}{2 + r}$$

where  $E[y_2] = \frac{1}{2}y_2^H + \frac{1}{2}y_2^L = \frac{1}{2}3 + \frac{1}{2}1 = 2$  denotes the expected second-period income.

c. Compare your answers to parts (a) and (b).

- In expectation, Andreu's savings when he is certain about his future income being  $y_2 = 1$  (in part a of the exercise) and when he is uncertain (but his expected income is  $E[y_2] = 2$ ) coincide. Intuitively, Andreu's savings take into account that his future income is, in expectation, 2.

2. Consider a cumulative distribution function  $F(x)$  which first-order stochastically dominates  $G(x)$ .

(a) Show that the mean of  $x$  under  $G(x)$ ,  $\int x dG(x)$ , cannot exceed that under  $F(x)$ ,  $\int x dF(x)$ , i.e.,

$$\int x dF(x) \geq \int x dG(x)$$

- We know that distribution function  $F(x)$  first-order stochastically dominates  $G(x)$  if

$$\int u(x) dF(x) \geq \int u(x) dG(x)$$

Using the fact that the utility function is weakly increasing, we have

$$\int x dF(x) \geq \int x dG(x)$$

(b) Provide now an example where  $\int x dF(x) \geq \int x dG(x)$  is satisfied, but  $F(x)$  does not first order stochastically dominates  $G(x)$ .

- Consider the two lotteries depicted in figure 1. The first one,  $F(x)$ , assigns 1/2 to monetary outcome \$2 and 1/2 to \$3. The second lottery,  $G(x)$ , evenly splits the probability weight that lottery  $F(x)$  assigns to \$2 between \$1 and \$2 (each occurring with a probability of 1/4), and the probability weight of

\$3 is also equally divided between \$3 and \$4.

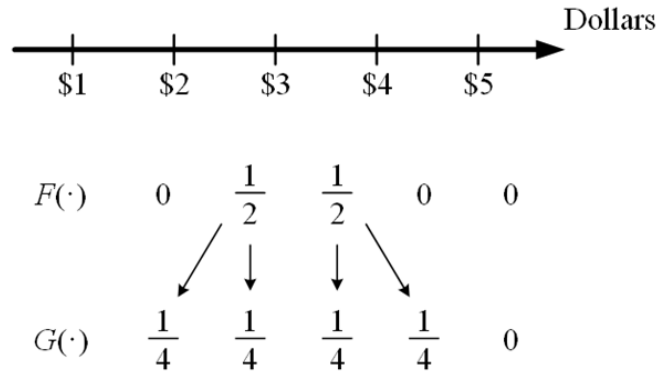


Figure 1. Lotteries  $F(x)$  and  $G(x)$ .

- Thus, both lotteries  $F(x)$  and  $G(x)$  have the same expected value,  $\frac{5}{2}$ . However, neither  $F(x)$  FOSD  $G(x)$ , nor  $G(x)$  FOSD  $F(x)$ . In particular, depicting both probability distributions (see figure 2), one can easily observe that lottery  $G(x)$  lies weakly above  $F(x)$  for outcomes  $x \leq \$3$ , but lies below for monetary outcomes beyond that threshold.

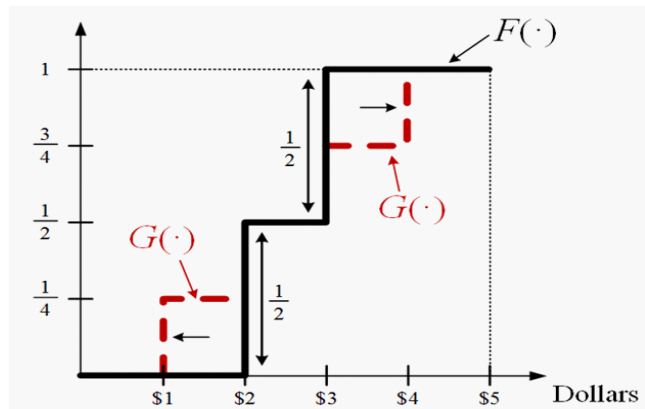


Figure 2. Lotteries  $F(x)$  and  $G(x)$ .