

# Recitation 11 (November 13th, 2020)

1. Consider an economy with two individuals  $i = \{1, 2\}$  with the following quasi-linear utility function

$$u_i(s^i, q^i) = v^i(s^i) + \alpha w^i$$

where  $s^i$  denotes the speed at which individual  $i$  drives his car,  $w^i$  is his wealth, and  $\alpha > 0$ . The utility that individual  $i$  obtains from driving fast is  $v^i(s^i)$ , which is increasing but concave in speed, whereby  $\frac{\partial v^i(s^i)}{\partial s^i} > 0$  and  $\frac{\partial^2 v^i(s^i)}{(\partial s^i)^2} < 0$ . Driving fast, however, increases the probability of suffering a car accident, represented by  $\gamma(s^i, s^j)$ . This probability is increasing both in the speed at which individual  $i$  drives,  $s^i$ , and the speed at which other individuals drive,  $s^j$ , where  $j \neq i$ . Hence, the speed of other individuals imposes a negative externality on driver  $i$ , since it increases his risk of suffering a car accident. If individual  $i$  suffers an accident, he bears a cost of  $c^i > 0$ , which intuitively embodies the cost of fixing his car, health-care expenses, etc.

- (a) *Unregulated equilibrium.* Set up individual  $i$ 's expected utility maximization problem. Take first-order conditions with respect to  $s^i$ , and denote the (implicit) solution to this first-order condition as  $\widehat{s}^i$ .

- With probability  $\gamma(s^i, s^j)$ , the individual suffers a car accident, and thus his utility is  $v^i(s^i) + \alpha w^i - c^i$ , and with probability  $1 - \gamma(s^i, s^j)$  he does not suffer the accident, leaving his utility level at  $v^i(s^i) + \alpha w^i$ .
- Hence, his expected utility is

$$\gamma(s^i, s^j)[v^i(s^i) + \alpha w^i - c^i] + (1 - \gamma(s^i, s^j))[v^i(s^i) + \alpha w^i],$$

which reduces to  $v^i(s^i) + \alpha w^i - \gamma(s^i, s^j)c^i$ . Hence, every individual  $i$  maximizes his expected utility by choosing an speed level  $s^i$  that solves

$$\max_{s^i} v^i(s^i) + \alpha w^i - \gamma(s^i, s^j) \times c^i$$

Taking first-order conditions with respect to  $s^i$  we obtain

$$\frac{\partial v^i(s^i)}{\partial s^i} - \frac{\partial \gamma}{\partial s^i} c^i = 0 \tag{1}$$

Hence, driver  $i$  independently selects the speed,  $\widehat{s}^i$ , that solves  $\frac{\partial v^i(s^i)}{\partial s^i} = \frac{\partial \gamma}{\partial s^i} c^i$ .

- Intuitively, driver  $i$  increases his speed  $s^i$  until the point where the additional utility from marginally increasing  $s^i$ ,  $\frac{\partial v^i(s^i)}{\partial s^i}$ , coincides with its associated ex-

pected individual cost from speed, i.e., a higher probability of suffering a car accident times its associated cost, as measured by  $\frac{\partial \gamma}{\partial s^i} c^i$ .

- *Parametric example.* Consider, for instance, a utility from driving fast of  $v(s^i) = \sqrt{s^i}$  (which is increasing and concave in  $s^i$ , as required), and that the probability of suffering a car accident is  $\gamma(s^i, s^j) = \beta_i s^i + \beta_j s^j$ , where  $\beta_i > \beta_j$  (indicating that my own speed increases the probability that I suffer a car accident more than other drivers' speeds). First order condition (1) in this context becomes

$$\frac{1}{2\sqrt{s^i}} = \beta_i c^i,$$

and solving for  $s^i$ , we obtain an equilibrium speed of  $\hat{s}^i = \frac{1}{4(\beta_i c^i)^2}$  for every individual driver  $i = \{1, 2\}$ .

- (b) *Social optimum.* Set up the social planner's expected welfare maximization problem. Take first-order conditions with respect to  $s^1$  and  $s^2$ . Denote the (implicit) solution to this first-order condition as  $\bar{s}^i$ .

- The social planner solves the expected welfare maximization problem

$$\max_{s^1, s^2} v^1(s^1) + \alpha w^1 + v^2(s^2) + \alpha w^2 - \gamma(s^1, s^2) \times (c^1 + c^2)$$

Taking first-order conditions with respect to  $s^1$ , we obtain that  $\bar{s}^1$  solves

$$\frac{\partial v^1(s^1)}{\partial s^1} = \frac{\partial \gamma}{\partial s^1} (c^1 + c^2) \quad (2)$$

and similarly with respect to  $s^2$ , we obtain that  $\bar{s}^2$  solves

$$\frac{\partial v^2(s^2)}{\partial s^2} = \frac{\partial \gamma}{\partial s^2} (c^1 + c^2) \quad (3)$$

Intuitively, at the social optimum every driver  $i$  increases his speed  $s^i$  until the point where the additional utility from marginally increasing  $s^i$  coincides with its associated expected social cost from speed, measured by not only the higher probability of him suffering a car accident but also by the higher probability that the other individual  $j \neq i$  suffers a car accident because of the speed  $s^i$  of individual  $i$ .

- (c) *Comparison.* Show that drivers have individual incentives to drive too fast, relative to the socially optimal speed, i.e., show that  $\hat{s}^i > \bar{s}^i$ .

- Comparing expressions (1) and (2), yields

$$\frac{\partial v^1(\hat{s}^1)}{\partial s^1} < \frac{\partial v^1(\bar{s}^1)}{\partial s^1}$$

Since  $\frac{\partial^2 v^i(s^i)}{(\partial s^i)^2} < 0$  by definition,  $\frac{\partial v^i(s^i)}{\partial s^i}$  is a decreasing function. Therefore, the speed that individual 1 independently selects,  $\hat{s}^1$ , is excessive from a social point of view, i.e.,  $\hat{s}^1 > \bar{s}^1$ . Similarly, comparing (1) and (3), we have that  $\hat{s}^2 > \bar{s}^2$ . Intuitively, every driver does not internalize the negative externality that his speed imposes on other drivers (in the form of a higher probability of suffering a car accident) when he independently selects his driving speed.

- Figure 1 represents the marginal utility,  $\frac{\partial v^i(s^i)}{\partial s^i}$ , and marginal expected costs, individual marginal costs,  $\frac{\partial \gamma}{\partial s^i} c^i$ , and social marginal costs,  $\frac{\partial \gamma}{\partial s^i} (c^i + c^j)$ , to support the above explanation. Since the social marginal cost curve is higher for any speed level  $s^i$  than the individual marginal cost curve, the former crosses the marginal utility curve at a lower speed level, i.e.,  $\bar{s}^i < \hat{s}^i$ . Intuitively, the social planner internalizes the externality that additional speed imposes on other drivers (who could suffer a car accident due to the speed of driver  $i$ ), and thus reduces the speed of both drivers.

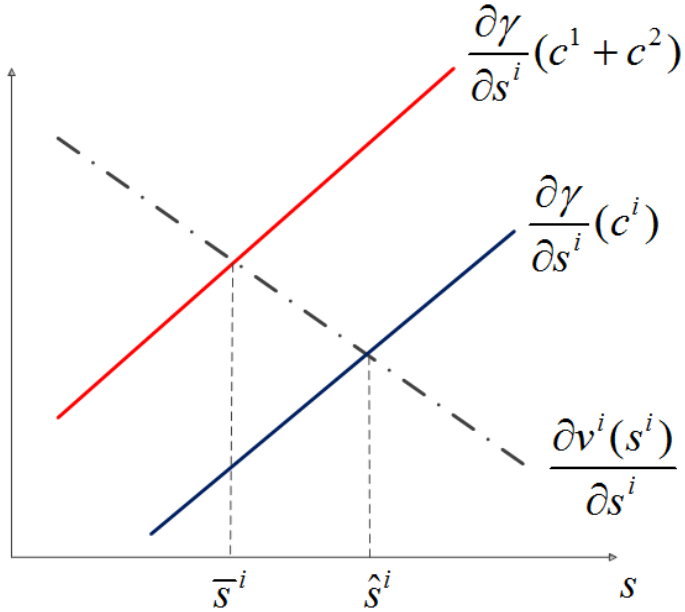


Figure 1. Efficient and socially optimal speed.

- Note that for simplicity, we consider that the marginal utility decreases in  $s^i$  at a constant rate, i.e.,  $\frac{\partial^2 v^i(s^i)}{\partial s^i{}^2}$  is constant in  $s^i$  or, alternatively,  $\frac{\partial^3 v^i(s^i)}{(\partial s^i)^3} = 0$ ; implying that the marginal utility curve is a straight line. In

addition, we also assume that further increases in speed  $s^i$  imply a constant increase in the probability of an accident, i.e.,  $\frac{\partial^2 \gamma}{(\partial s^i)^2} > 0$  but constant or, alternatively, that  $\frac{\partial^3 \gamma}{(\partial s^i)^3} = 0$ . This property entails the marginal cost curve is also a straight line.

- *Parametric example.* Continuing with the previous example in which  $v^i(s^i) = \sqrt{s^i}$  and  $\gamma(s^i, s^j) = \beta_i s^i + \beta_j s^j$ , the socially optimal speed that the social planner would select,  $\bar{s}^i$ , is that satisfying

$$\frac{1}{2\sqrt{\bar{s}^i}} = \beta_i(c^i + c^j)$$

and solving for  $\bar{s}^i$  yields  $\bar{s}^i = \frac{1}{4[\beta_i(c^i + c^j)]^2}$ , which clearly falls below the speed level independently selected by every driver  $\hat{s}^i = \frac{1}{4(\beta_i c^i)^2}$ .

- (d) *Restoring the social optimum.* Let us now evaluate the effect of speeding tickets (fines) to individuals driving too fast, i.e., to those drivers with a speed  $\hat{s}^i$  satisfying,  $\hat{s}^i > \bar{s}^i$ . What is the dollar amount of the fine  $m^i$  that induces every individual  $i$  to fully internalize the externality he imposes onto others?

- Comparing (1) and (2) for driver 1, we must impose a fine of  $m^1 = c^2$  in order to guarantee that (1) coincides with (2). Intuitively, this fine induces driver 1 to internalize the negative externality (higher chances of suffering a car accident and, in this case, an associated monetary cost of repairs) that he imposes on driver 2. Similarly comparing (1) and (3) for driver 2, we must impose a fine of  $m^2 = c^1$  in order to guarantee that (1) coincides with (3).

- (e) Let us now consider that individuals obtain a utility from driving fast,  $v^i(s^i)$ , only in the case that no accident occurs. Repeat steps (a)-(c), finding the optimal fine  $m^i$  that induces individuals to fully internalize the externality.

- *Equilibrium speed.* In this section of the exercise, driver  $i$  only obtains utility from driving fast,  $v^i(s^i)$ , when no accident occurs. Given that the probability that an accident does *not* occur is  $1 - \gamma(s^1, s^2)$ , the utility of driver  $i$  is

$$\underbrace{[1 - \gamma(s^1, s^2)] (v^i(s^i) + \alpha w^i)}_{\text{No accident}} + \underbrace{\gamma(s^i, s^j)(\alpha w^i - c^i)}_{\text{Accident}}$$

which can be rearranged as

$$v^i(s^i) + \alpha w^i - \gamma(s^i, s^j) [c^i + v^i(s^i)]$$

Taking first order conditions with respect to  $s^i$ , we obtain that the individual

driver  $i$  independently selects the speed  $\bar{s}^i$  that solves

$$\frac{\partial v^i(s^i)}{\partial s^i} [1 - \gamma(s^i, s^j)] = \frac{\partial \gamma}{\partial s^i} (c^i + v^i(s^i)) \quad (4)$$

where conveniently separates the marginal utility of driving faster in the left-hand side, which only arises if driver  $i$  does not suffer a car accident, an event with probability  $1 - \gamma(s^i, s^j)$ ; and its associated marginal cost in the right-hand side, which captures the higher probability of suffering a car accident,  $\frac{\partial \gamma}{\partial s^i}$ , and its two costs: one explicit,  $c^i$ , and one implicit, namely, the utility from driving that driver  $i$  would have to give up (since he can only benefit from driving when he does not suffer a car accident).

– *Parametric example.* Following with the on-going parametric example, the above first order condition (4) becomes

$$\frac{1}{2\sqrt{\bar{s}^i}} [1 - (\beta_i \bar{s}^i + \beta_j \bar{s}^j)] = \beta_i (c^i + \sqrt{\bar{s}^i}),$$

and similarly for driver  $j$ . Before solving for  $\bar{s}^i$  in order to driver  $i$ 's best response function, let us assume (in order to keep our parametric example compact) that  $\beta_i = \beta_j = \frac{1}{2}$  and  $c^i = c^j = \frac{2}{3}$ . In this context, solving for  $\bar{s}^i$  we obtain

$$\bar{s}^i(\bar{s}^j) = 2 - 3\bar{s}^j - \frac{4}{3}\sqrt{\bar{s}^j}$$

Since both drivers are symmetric,  $\bar{s}^i = \bar{s}^j$ , we can solve for  $\bar{s}^i$  yielding a symmetric equilibrium speed level of  $\bar{s}^i = 0.313$ .

- *Socially optimal speed.* The social planner's maximization problem in this case becomes

$$\max_{s^1, s^2} v^1(s^1) + \alpha w^1 - \gamma(s^1, s^2) [c^1 + v^1(s^1)] + v^2(s^2) + \alpha w^2 - \gamma(s^1, s^2) [c^2 + v^2(s^2)]$$

Taking first order conditions with respect to  $s^i$ , we obtain that the socially optimal speed,  $\tilde{s}^i$ , solves

$$\frac{\partial v^i(s^i)}{\partial s^i} [1 - \gamma(s^i, s^j)] = \frac{\partial \gamma}{\partial s^i} [c^i + v^i(s^i)] + \frac{\partial \gamma}{\partial s^i} [c^j + v^j(s^j)] \quad (5)$$

- *Comparison.* Comparing expressions (4) and (5), we obtain that the fine  $m^i$  that induces every individual  $i$  to internalize the externality that his driving imposes on others is

$$m^i = c^j + v^j(s^j)$$

Intuitively, now an increase in the speed of driver  $i$  not only increases the probability that driver  $j$  suffers a car accident, and thus needs to incur a cost of  $c^j$ , it also reduces the utility from driving that driver  $j$  can only experience if he is not involved in a car accident.

- *Parametric example.* Following with the on-going parametric example, the above first order condition (4) becomes

$$\frac{1}{2\sqrt{\widehat{s}^i}} [1 - (\beta_i \widehat{s}^i + \beta_j \widehat{s}^j)] = \beta_i [(c^i + \sqrt{\widehat{s}^i}) + (c^j + \sqrt{\widehat{s}^j})],$$

Before solving for  $\widehat{s}^i$  in order to driver  $i$ 's best response function, let us assume (in order to keep our parametric example compact) that  $\beta_i = \beta_j = \frac{1}{2}$  and  $c^i = c^j = \frac{2}{3}$ . In this context, we can simultaneously solve for  $\widehat{s}^i$  and  $\widehat{s}^j$  obtaining  $\widehat{s}^i = \widehat{s}^j = 0.157$ , which is indeed a lower speed than when drivers independently choose their own driving speed,  $\bar{s}^i = 0.313$ .