EconS 501 Midterm #2 - November 18th, 2020

Show all your work clearly and make sure you justify all your answers.
NAME________________________

1. Consider an exponentially distributed event (e.g., machine breakdown, or receiving a junk email) occurring at a rate λ before period x, where $\lambda > 0$ and $x \geq 0$, with the following cumulative distribution function (CDF):

$$F(x;\lambda) = 1 - \exp(-\lambda x)$$

   (a) Show that if $\lambda_1 > \lambda_2$, then $F(x;\lambda_2)$ first order stochastically dominates $F(x;\lambda_1)$.

   - Differentiating $F(x;\lambda)$ with respect to λ, we have

$$F_\lambda(x;\lambda) = x\exp(-\lambda x) > 0$$

     implying that $F(x;\lambda)$ is monotonically increasing in λ. Therefore, if $\lambda_1 > \lambda_2$, $F(x;\lambda_1) > F(x;\lambda_2)$ for all $x \geq 0$. Intuitively, a higher rate of occurrence means that the event is more likely to occur under λ_1, within any period x, than under λ_2, so that $F(x;\lambda_2)$ first order stochastically dominates $F(x;\lambda_1)$.

   (b) Define the hazard rate to be

$$HR(x;\lambda) = \frac{f(x;\lambda)}{1 - F(x;\lambda)}$$

   where $f(x;\lambda) = F_x(x;\lambda)$ is the probability distribution function (PDF) of an event.

   Show that if $\lambda_1 > \lambda_2$, $F(x;\lambda_2)$ dominates $F(x;\lambda_1)$ in terms of hazard rate; that is, $HR(x;\lambda_1) > HR(x;\lambda_2)$. Interpret your results.

   - The hazard rate of the exponentially distributed event is

$$\begin{aligned} HR(x;\lambda) &= \frac{f(x;\lambda)}{1 - F(x;\lambda)} \\ &= \frac{\lambda\exp(-\lambda x)}{\exp(-\lambda x)} \\ &= \lambda \end{aligned}$$

     Thus, $HR(x;\lambda_1) > HR(x;\lambda_2)$ if $\lambda_1 > \lambda_2$. Intuitively, the event occurs more frequently within period x (with a higher hazard rate) under rate λ_1 than under rate λ_2, so that $F(x;\lambda_2)$ dominates $F(x;\lambda_1)$ in terms of hazard rate.

   (c) Define the reverse hazard rate to be

$$RHR(x;\lambda) = \frac{f(x;\lambda)}{F(x;\lambda)}$$

   Show that if $\lambda_1 > \lambda_2$, $F(x;\lambda_2)$ dominates $F(x;\lambda_1)$ in terms of reverse hazard rate; that is, $RHR(x;\lambda_2) > RHR(x;\lambda_1)$. Interpret your results.

- The reverse hazard rate of the exponentially distributed event is

$$RHR(x;\lambda) = \frac{f(x;\lambda)}{F(x;\lambda)} = \frac{\lambda \exp(-\lambda x)}{1-\exp(-\lambda x)}$$

Differentiating $RHR(x;\lambda)$ with respect to λ, we obtain

$$\frac{\partial RHR(x;\lambda)}{\partial \lambda} = \frac{[\exp(-\lambda x) - \lambda x \exp(-\lambda x)][1-\exp(-\lambda x)] - x\lambda [\exp(-\lambda x)]^2}{[1-\exp(-\lambda x)]^2} = \frac{\exp(-\lambda x)}{[1-\exp(-\lambda x)]^2}[1-\lambda x - \exp(-\lambda x)]$$

Define $g(x;\lambda) \equiv 1-\lambda x - \exp(-\lambda x)$, and differentiate with respect to x,

$$\frac{\partial g(x;\lambda)}{\partial x} = -\lambda + \lambda \exp(-\lambda x) = -\lambda[1-\exp(-\lambda x)] < 0$$

so that $g(x;\lambda)$ attains a maximum at $x=0$, at which

$$g(0;\lambda) = 1-0-\exp(0) = 0.$$

Substituting into the partial derivative of the reverse hazard rate function, yields

$$\frac{\partial RHR(x;\lambda)}{\partial \lambda} < \frac{\exp(-\lambda x)}{[1-\exp(-\lambda x)]^2}\underbrace{g(0;\lambda)}_{=0} = 0$$

so that the reverse hazard rate, $RHR(x;\lambda)$, is monotonically decreasing in λ.

Therefore, we can conclude that $RHR(x;\lambda_2) > RHR(x;\lambda_1)$ if $\lambda_1 > \lambda_2$. Intuitively, the event not occurring before period x is more likely under rate λ_2 than under rate λ_1, so that $F(x;\lambda_2)$ dominates $F(x;\lambda_1)$ in terms of reverse hazard rate.

(d) Define the likelihood ratio to be

$$LR(x;\lambda_1,\lambda_2) = \frac{f(x;\lambda_2)}{f(x;\lambda_1)}$$

Show that if $\lambda_1 > \lambda_2$, $F(x;\lambda_2)$ dominates $F(x;\lambda_1)$ in terms of likelihood ratio; that is, $LR(x;\lambda_1,\lambda_2) < LR(y;\lambda_1,\lambda_2)$ for all $x<y$. Interpret your results.

- The likelihood ratio of the exponentially distributed event is

$$LR(x;\lambda_1,\lambda_2) = \frac{f(x;\lambda_2)}{f(x;\lambda_1)} = \frac{\lambda_2 \exp(-\lambda_2 x)}{\lambda_1 \exp(-\lambda_1 x)} = \frac{\lambda_2}{\lambda_1}\exp((\lambda_1-\lambda_2)x)$$

When $x < y$, let us compare the likelihood of the event occurring by checking whether $LR(x;\lambda_1,\lambda_2) < LR(y;\lambda_1,\lambda_2)$ holds or not, that simplifies to

$$\frac{\lambda_2}{\lambda_1}\exp\left(\left(\lambda_1-\lambda_2\right)x\right) < \frac{\lambda_2}{\lambda_1}\exp\left(\left(\lambda_1-\lambda_2\right)y\right)$$

and further reduces to

$$\exp\left(-\left(\lambda_1-\lambda_2\right)\left(y-x\right)\right) < 1$$

Taking natural algorithm on both sides of the above expression, we obtain

$$\left(\lambda_1-\lambda_2\right)\left(y-x\right) > 0$$

which holds because $y > x$ and $\lambda_1 > \lambda_2$ by definition.
Hence, we can conclude htat $LR(x;\lambda_1,\lambda_2) < LR(y;\lambda_1,\lambda_2)$ if $\lambda_1 > \lambda_2$. Intuitively, the event occurring under rate λ_1 is more (less) likely than under rate λ_2 at a shorter (longer) period x (y), so that $F(x;\lambda_2)$ dominates $F(x;\lambda_1)$ in terms of likelihood ratio.

(e) What can you infer from the results in parts (a) to (d)? Explain.

- From the above results, we can infer that if $F(x;\lambda_2)$ first order stochastically dominates $F(x;\lambda_1)$, $F(x;\lambda_2)$ also dominates $F(x;\lambda_1)$ in terms of hazard rate, reverse hazard rate, and likelihood ratio. In this regard, these four measures of stochastic dominance are equivalent to each other, and summarized as follows:

$$\text{FOSD} \Leftrightarrow \text{HRD} \Leftrightarrow \text{RHRD} \Leftrightarrow \text{LRD}$$

2. Consider an individual with initial wealth w, where $w > 0$, purchasing car insurance to insure against the loss from a car accident (such as car wreck, physical damages, body injury, and lost income, etc.).

   - When a car accident does not occur with probability π_1, his income is $x_1 = w + z_1$ where z_1 represents the income he receives from the insurance company in state 1.
   - When a car accident occurs with probability π_2, his income is $x_2 = w - D + z_2$, where $D > 0$ represents the monetary loss from the car accident and z_2 is the income he receives from the insurance company in state 2.

   Assume that the insurance company operates in a perfectly competitive market and the policy is actuarially fair satisfying $\pi_1 z_1 + \pi_2 z_2 = 0$, where $\pi_1 + \pi_2 = 1$ and $z_2 > 0 > z_1$, indicating that he pays an insurance premium without an accident and receives an indemnity when an accident occurs. Let $u(x_i)$ be his utility, where $i \in \{1, 2\}$, when state i occurs, and assume that $u(0) = 0$, $u'(x_i) > 0$, and $u''(x_i) < 0$.

   (a) Show that the individual will fully insure against the risk of a car accident.

- The individual chooses z_1 and z_2 to solve the following expected utility maximization problem,

$$\max_{z_1,z_2\geq 0} u\left(z_1, z_2; \pi_1, \pi_2\right) = \pi_1 u\left(w + z_1\right) + \pi_2 u\left(w - D + z_2\right)$$

$$\text{subject to} \quad \pi_1 z_1 + \pi_2 z_2 = 0$$

The Lagrangian function is

$$L = \pi_1 u\left(w + z_1\right) + \pi_2 u\left(w - D + z_2\right) - \lambda\left(\pi_1 z_1 + \pi_2 z_2\right)$$

Taking the first order condition with respect to z_1 and z_2, we obtain

$$\frac{\partial L}{\partial z_1} = \pi_1 u'\left(w + z_1\right) - \lambda \pi_1 \leq 0$$
$$\frac{\partial L}{\partial z_2} = \pi_2 u'\left(w - D + z_2\right) - \lambda \pi_2 \leq 0$$

with the associated Kuhn-Tucker conditions

$$z_1 \frac{\partial L}{\partial z_1} = 0$$
$$z_2 \frac{\partial L}{\partial z_2} = 0$$

Assuming interior solutions, where $\frac{\partial L}{\partial z_1} = \frac{\partial L}{\partial z_2} = 0$, we have

$$\pi_1 u'\left(w + z_1\right) = \lambda \pi_1$$
$$\pi_2 u'\left(w - D + z_2\right) = \lambda \pi_2$$

Dividing the first expression by the second, yields

$$\frac{\pi_1 u'\left(w + z_1\right)}{\pi_2 u'\left(w - D + z_2\right)} = \frac{\pi_1}{\pi_2}$$

Cancelling out $\frac{\pi_1}{\pi_2}$ on both sides of the above expression,

$$u'\left(w + z_1\right) = u'\left(w - D + z_2\right)$$

Since $u' > 0$, monotonicity of the utility function entails

$$\underbrace{w + z_1}_{=x_1} = \underbrace{w - D + z_2}_{=x_2}$$

so that as the final wealth of the individual is the same whether the car accident occurs or not, the individual will fully insure against the risk of car accident that is illustrated in Figure 2 as the tangency between the black indifference curve and the zero-profit line that intersect at the 45^o degree line where $x_1 = x_2$.

(b) Assume that the individual assigns subjective probability of $\pi_1+\varepsilon$ to a car accident not occurring in state 1 and $\pi_2 - \varepsilon$ to a car accident occurring in state 2. Show that this individual will underinsure when $\varepsilon > 0$ but overinsure when $\varepsilon < 0$.

- As the marginal rate of substitution equals the price ratio, we have

$$\frac{(\pi_1 + \varepsilon)\, u'(w + z_1)}{(\pi_2 - \varepsilon)\, u'(w - D + z_2)} = \frac{\pi_1}{\pi_2}$$

Rearranging, we obtain

$$\frac{u'(w + z_1)}{u'(w - D + z_2)} = \frac{\pi_1}{\pi_1 + \varepsilon} \frac{\pi_2 - \varepsilon}{\pi_2}$$

When $\varepsilon > 0$, the right hand side becomes

$$\underbrace{\frac{\pi_1}{\pi_1 + \varepsilon}}_{<1} \underbrace{\frac{\pi_2 - \varepsilon}{\pi_2}}_{<1},$$

so that

$$u'(w + z_1) < u'(w - D + z_2)$$

Since $u'' < 0$, the above inequality can be rewritten as

$$\underbrace{w + z_1}_{=x_1} > \underbrace{w - D + z_2}_{=x_2}$$

so that the final wealth of the individual when an accident does not occur (x_1) exceeds his final wealth when an accident occurs (x_2), and he underinsures against the risk of a car accident when $\varepsilon > 0$.
An opposite argument applies when $\varepsilon < 0$, in which

$$\underbrace{w + z_1}_{=x_1} < \underbrace{w - D + z_2}_{=x_2}$$

entails that the final wealth of the individual when an accident does not occur (x_1) falls below his final wealth when an accident occurs (x_2), so that this individual overinsures against the risk of a car accident when $\varepsilon < 0$.

(c) Suppose $\varepsilon(D)$ decreases in D. What happen to the individual's insurance premium z_1 and indemnity payment z_2 when damages from the car accident become more significant (D increases)? Consider the case of $\varepsilon > 0$ only, and explain your results.

- The equilibrium condition for an interior optimum can be rewritten as

$$\frac{\pi_1 + \varepsilon(D)}{\pi_2 - \varepsilon(D)} \frac{u'(w + z_1)}{u'(w - D + z_2)} = \frac{\pi_1}{\pi_2}$$

When D increases, $\varepsilon(D)$ decreases, so that the fraction $\frac{\pi_1+\varepsilon(D)}{\pi_2-\varepsilon(D)}$ decreases, meaning that the individual places a higher (lower) subjective probability on the accident (not) occurring as the damages become more significant.

- For the equilibrium condition to hold, we need the fraction $\frac{u'(w+z_1)}{u'(w-D+z_2)}$ to increase so that its multiple with $\frac{\pi_1+\varepsilon(D)}{\pi_2-\varepsilon(D)}$ stays the same at $\frac{\pi_1}{\pi_2}$ on the right-hand side of the above expression. This means the marginal utility $u'(w+z_1)$ in the numerator must increase and $u'(w-D+z_2)$ in the denominator must decrease. By the decreasingness of the marginal utility function, where $u''<0$, this implies $x_1 = w + z_1$ must decrease and $x_2 = w - D + z_2$ must increase. which holds only if the individual pays a higher insurance premium (z_1 becoming more negative) in exchange for a higher indemnity payment (z_2 becoming more positive).

3. Consider two individuals with utility functions over two goods, x and y, given by

$$\begin{aligned} U_1 &= \log(x_1) + y_1 - \frac{1}{2}\log(x_2) \quad \text{for consumer 1, and} \\ U_2 &= \log(x_2) + y_2 - \frac{1}{2}\log(x_1) \quad \text{for consumer 2.} \end{aligned}$$

where the consumption of good x by individual $i = \{1,2\}$ creates a negative externality on individual $j \neq i$ (see the third term, which enters negatively on each individual's utility function). For simplicity, consider that both individuals have the same wealth, w, and that the price for both goods is 1.

(a) *Unregulated equilibrium.* Set up consumer 1's utility maximization problem, and determine his demand for goods x and y, as x_1 and y_1. Then operate similarly to find consumer 2's demand for good x and y, as x_2 and y_2.

- Consumer A chooses x_1 and y_1 to solve

$$\max_{(x_1,y_1)} \ \log(x_1) + y_1 - \frac{1}{2}\log(x_2)$$
$$\text{subject to } \ x_1 + y_1 = w$$

The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_2) + \lambda(w - x_1 - y_1),$$

which yields first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y_1} = 1 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = w - x_1 - y_1 = 0$$

Solving for x_1, we obtain $\frac{1}{x_1} = 1$, i.e., $x_1 = 1$, which implies $w - 1 - y_1 = 0$, or $y_1 = w - 1$. Hence, consumer 1's optimal consumption is

$$x_1 = 1 \ \text{ and } \ y_1 = w - 1$$

A similar argument applies to consumer 2,

$$x_2 = 1 \ \text{ and } \ y_2 = w - 1$$

(b) *Social optimum.* Calculate the socially optimal amounts of x_1, y_1, x_2 and y_2, considering that the social planner maximizes a utilitarian social welfare function, namely, $W = U_1 + U_2$.

- The socially optimal consumption in this case solves

$$\max_{(x_1,y_1)} U_1 + U_2 \text{ subject to } x_1 + y_1 = w \text{ and } x_2 + y_2 = w$$

The Lagrangian for this social planner's problem is

$$\mathcal{L} = \frac{1}{2}\log(x_1) + \frac{1}{2}\log(x_2) + y_1 + y_2 + \lambda_1(w - x_1 - y_1) + \lambda_2(w - x_2 - y_2)$$

Taking first-order conditions, we find the socially optimal consumption profile:

$$x_1 = \frac{1}{2} \text{ and } y_1 = w - \frac{1}{2}$$

$$x_2 = \frac{1}{2} \text{ and } y_2 = w - \frac{1}{2}$$

Intuitively, the social planner recommends a lower consumption of good x (the good that generates the negative externality), and an increase in the consumption of good y, for both individuals.

(c) *Restoring efficiency.* Show that the social optimum you found in part (b) can be induced by a tax on good x (so the after-tax price becomes $1+t$) with the revenue returned equally to both consumers in a lump-sum transfer.

- With tax t placed on good x and with lump-sum transfer T, consumer 1 solves

$$\max_{(x_1,y_1)} \log(x_1) + y_1 - \frac{1}{2}\log(x_2)$$
$$\text{subject to } (1+t)x_1 + y_1 = w + T$$

where note that the price of good x increased from 1 to $(1+t)$, but this consumer also sees his wealth increase by the lump sum T. The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1) + y_1 - \frac{1}{2}\log(x_2) \; + \lambda(w + T - \; (1+t)x_1 - y_1)$$

Taking first-order conditions, we obtain

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda(1+t) = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = 1 - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w + T - (1+t)x_1 - y_1 = 0$$

Simultaneously solving for x_1 and y_1, we find that consumer 1's consumption bundles after introducing the tax become

$$x_1 = \frac{1}{1+t} \text{ and } y_1 = w + T - 1$$

Similarly we find the optimal consumption of consumer 2 who pays tax t on good x and receives Tas a lump-sum transfer:

$$x_2 = \frac{1}{1+t} \text{ and } y_2 = w + T - 1$$

- *Comparison.* Comparing the optimal consumption levels found in part (b) with the equilibrium outcomes found in part (c), the tax imposed on any individual must hence satisfy

$$\frac{1}{2} = \frac{1}{1+t},$$

which would guarantee that equilibrium and socially optimal amounts coincide. Solving for the tax t yields $t = \$1$. Hence, by setting a tax of $t = \$1$ on the consumption of good x, and returning the tax revenue to this individual in a lump-sum transfer, efficiency is restored, yielding a consumption

$$x_1 = \frac{1}{1+1} = \frac{1}{2} \text{ of good } x,$$

and

$$\begin{aligned} y_1 &= w + T - 1 \\ &= M + \frac{1}{2} - 1 = M - \frac{1}{2} \text{ of good } y, \end{aligned}$$

as described in the socially optimal amounts found in part (b).

4. Consider a competitive market in which the government will be imposing an ad valorem tax of τ. Aggregate demand curve is $x(p) = \frac{A}{\sqrt{p}}$, where $A > 0$, and aggregate supply curve $q(p) = \alpha p^{\frac{1}{4}}$, where $\alpha > 0$. Denote $\lambda = (1+\tau)$. Assume that a partial equilibrium analysis is valid.

   (a) Evaluate how the equilibrium price is affected by a marginal increase in the tax, i.e., a marginal increase in λ.

Imposing the ad valorem tax τ, the aggregate demand curve now becomes

$$x(p,\tau) = A\left[p(\tau)(1+\tau)\right]^{-\frac{1}{2}}$$

Whereas, the aggregate supply curve becomes

$$q(p,\tau) = \alpha\left[p(\tau)\right]^{\frac{1}{4}}$$

At equilibrium, the aggregate supply curve intersects the aggregate demand curve so that the market clears, that is,

$$A\left[p\left(\tau\right)\left(1+\tau\right)\right]^{-\frac{1}{2}}=\alpha\left[p\left(\tau\right)\right]^{\frac{1}{4}}$$

Rearranging, we can express the price $p\left(\tau\right)$ that the producers receive in terms of tax rate τ.

$$p\left(\tau\right)=\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}\left(1+\tau\right)^{-\frac{2}{3}}$$

Differentiating the above expression with respect to τ,

$$p'\left(\tau\right)=-\frac{2}{3}\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}\left(1+\tau\right)^{-\frac{5}{3}}$$

We evaluate the first order condition at $\tau=0$ for a marginal increase in the tax, so that

$$p'\left(0\right)=-\frac{2}{3}\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}\left(1+0\right)^{-\frac{5}{3}}$$

so that the tax burden borne by the producers is $p'\left(0\right)=-\frac{2}{3}c$, where $c\equiv\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}$ is a constant independent of τ. Essentially, c captures the responsiveness of the equilibrium price to the steepness of the aggregate demand and supply curves. In fact, for a higher A which represents a steeper aggregate demand curve, the price received by the producer is more responsive to a marginal change in the ad valorem tax τ, and the opposite is said for a higher α which represents a steeper aggregate supply curve.

We are also interested in the price paid by the consumers, given by

$$\hat{p}\left(\tau\right)\equiv p\left(\tau\right)\cdot\left(1+\tau\right)=\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}\left(1+\tau\right)^{-\frac{2}{3}}\left(1+\tau\right)=\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}\left(1+\tau\right)^{\frac{1}{3}}$$

Differentiating the above expression with respect to τ,

$$\hat{p}'\left(\tau\right)=\frac{1}{3}\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}\left(1+\tau\right)^{-\frac{2}{3}}$$

We evaluate the first order condition at $\tau=0$ for a marginal increase in the tax, so that

$$\hat{p}'\left(0\right)=\frac{1}{3}\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}$$

so that the tax burden borne by the consumers is $\hat{p}'\left(0\right)=\frac{1}{3}c$, where $c\equiv\left(\frac{A}{\alpha}\right)^{\frac{4}{3}}$.

(b) Describe the incidence of the tax when the supply is perfectly inelastic.

$$p'\left(\tau=0|\gamma=0\right)=\left(\frac{A}{\alpha}\right)^{2}\frac{1}{\left(1+\tau\right)}$$

$$\hat{p}'\left(\tau=0|\gamma=0\right)=\left(\frac{A}{\alpha}\right)^{2}$$

Therefore, when the supply is perfectly inelastic, the tax is entirely borne by the producers, and the price received by each one of them is reduced by a factor of $\left(\frac{A}{\alpha}\right)^2$ on the ad valorem tax τ. Whereas, the price paid by the consumers is unaffected by τ.

(c) What is the tax incidence when the demand is perfectly inelastic?

$$\begin{aligned}
p'(\tau = 0|\varepsilon = 0) &= \left(\frac{A}{\alpha}\right)^4 \\
\hat{p}'(\tau = 0|\varepsilon = 0) &= 0
\end{aligned}$$

Therefore, when the demand is perfectly inelastic, the tax is entirely borne by the consumers, and the price paid by each one of them is increased by a factor of $\left(\frac{A}{\alpha}\right)^4$ on the ad valorem tax τ. Whereas, the price received by the producers is unaffected by τ.

(d) What happens when each of these elasticities approaches ∞ in absolute value?

$$\begin{aligned}
\lim_{\gamma\to\infty} p'(0) &= \lim_{\gamma\to\infty}\left[\left(\frac{A}{\alpha}\right)^{\frac{1}{\gamma-\varepsilon}} \frac{\varepsilon}{\gamma-\varepsilon}\right] \\
&= \left(\lim_{\gamma\to\infty}\left(\frac{A}{\alpha}\right)^{\frac{1}{\gamma-\varepsilon}}\right)\cdot\left(\lim_{\gamma\to\infty}\frac{\varepsilon}{\gamma-\varepsilon}\right) \quad \text{by the continuity in the limits} \\
&= 1\cdot 0 = 0 \\
\lim_{\gamma\to\infty} \hat{p}'(0) &= \lim_{\gamma\to\infty}\left[\left(\frac{A}{\alpha}\right)^{\frac{1}{\gamma-\varepsilon}} \frac{\gamma}{\gamma-\varepsilon}\right] \\
&= \left(\lim_{\gamma\to\infty}\left(\frac{A}{\alpha}\right)^{\frac{1}{\gamma-\varepsilon}}\right)\left(\lim_{\gamma\to\infty}\frac{1}{1-\frac{\varepsilon}{\gamma}}\right) \quad \text{by the continuity in the limits} \\
&= 1\cdot 1 = 1
\end{aligned}$$

Therefore, when the supply is perfectly elastic, as given by $\gamma\to\infty$, the tax is entirely borne by the consumers by a factor of 1, and the price received by the producers is unaffected by the ad valorem tax τ.

$$\begin{aligned}
\lim_{\varepsilon\to-\infty} p'(0) &= \lim_{\varepsilon\to-\infty}\left[\left(\frac{A}{\alpha}\right)^{\frac{1}{\gamma-\varepsilon}} \frac{\varepsilon}{\gamma-\varepsilon}\right] \\
&= \left(\lim_{\varepsilon\to-\infty}\left(\frac{A}{\alpha}\right)^{\frac{1}{\gamma-\varepsilon}}\right)\cdot\left(\lim_{\varepsilon\to-\infty}\frac{1}{\frac{\gamma}{\varepsilon}-1}\right) \quad \text{by the continuity in the limits} \\
&= -1\cdot 1 = 1 \\
\lim_{\varepsilon\to-\infty} \hat{p}'(0) &= \lim_{\varepsilon\to-\infty}\left[\left(\frac{A}{\alpha}\right)^{\frac{1}{\gamma-\varepsilon}} \frac{\gamma}{\gamma-\varepsilon}\right] \\
&= \left(\lim_{\varepsilon\to-\infty}\left(\frac{A}{\alpha}\right)^{\frac{1}{\gamma-\varepsilon}}\right)\left(\lim_{\varepsilon\to-\infty}\frac{\gamma}{\gamma-\varepsilon}\right) \quad \text{by the continuity in the limits} \\
&= 1\cdot 0 = 0
\end{aligned}$$

Therefore, when the demand is perfectly elastic, as given by $\varepsilon \to -\infty$, the tax is entirely borne by the producers by a factor of 1, and the price received by the consumers is unaffected by the ad valorem tax τ.