Homework # 8 - [Due on Wednesday November 11th, 2020]

1. A tax is to be levied on a commodity bought and sold in a competitive market. Two possible forms of tax may be used: In one case, a *per unit* tax is levied, where an amount t is paid per unit bought or sold. In the other case, an *ad valorem* tax is levied, where the government collects a tax equal to τ times the amount the seller receives from the buyer. Assume that a partial equilibrium approach is valid.

   (a) Show that, with a per unit tax, the ultimate cost of the good to consumers and the amounts purchased are independent of whether the consumers or the producers pay the tax. As a guidance, let us use the following steps:

      1. *Consumers:* Let p^c be the competitive equilibrium price when the *consumer* pays the tax. Note that when the consumer pays the tax, he pays $p^c + t$ whereas the producer receives p^c. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the consumer pays the tax.

         - If the per unit tax t is levied on the consumer, then he pays $p+t$ for every unit of the good, and the demand at market price p becomes $x(p+t)$. The equilibrium market price p^c is determinded from equalizing demand and supply:

$$x\left(p^c + t\right) = q\left(p^c\right).$$

      2. *Producers:* Let p^p be the competitive equilibrium price when the *producer* pays the tax. Note that when the producer pays the tax, he receives $p^p - t$ whereas the consumer pays p^p. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the producer pays the tax.

         - On the other hand, if the per unit tax t is levied on the producer, then he collects $p - t$ from every unit of the good sold, and the supply at market price p becomes $q(p-t)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x\left(p^p\right) = q\left(p^p - t\right).$$

(b) Show that if an equilibrium price p solves your equality in part (a), then $p+t$ solves the equality in (b). Show that, as a consequence, equilibrium amounts are independent of whether consumers or producers pay the tax.

- It is easy to see that p solves the first equation if and only if $p+t$ solves the second one. Therefore, $p^p = p^c + t$, which is the ultimate cost of the good to consumers in both cases. The amount purchased in both cases is

$$x\left(p^p\right) = x\left(p^c + t\right).$$

(c) Show that the result in part (b) is not generally true with an ad valorem tax. In this case, which collection method leads to a higher cost to consumers? [*Hint:* Use the same steps as above, first for the consumer and then for the producer, but taking into account that now the tax increases the price to $(1+\tau)p$. Then, construct the excess demand function for the case of the consumer and the producer.]

- If the ad valorem tax τ is levied on the consumer, then he pays $(1+\tau)\,p$ for every unit of the good, and the demand at market price p becomes $x\left((1+\tau)\,p\right)$. The equilibrium market price p^c is determined from equalizing demand and supply:

$$x\left((1+\tau)\,p^c\right) = q\left(p^c\right).$$

On the other hand, if the ad valorem tax τ is levied on the producer, he receives $(1+\tau)\,p$ for every unit of the good sold, and the supply at market price p becomes $q\left((1-\tau)\,p\right)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x\left(p^p\right) = q\left((1-\tau)\,p^p\right).$$

Consider the excess demand function for this case:

$$z\left(p\right) = x\left(p\right) - q\left((1-\tau)\,p\right) \tag{1}$$

Since the demand curve $x\left(\cdot\right)$ is non-increasing and the supply curve $q\left(\cdot\right)$ is non-decreasing, $z\left(p\right)$ must be non-increasing. From (1) we have

$$\begin{aligned} z\left((1+\tau)\,p^c\right) &= x\left((1+\tau)\,p^c\right) - q\left((1-\tau)\left[(1+\tau)\,p^c\right]\right) = \\ &= x\left((1+\tau)\,p^c\right) - q\left(\left(1-\tau^2\right)p^c\right) \geq \\ &\geq x\left((1+\tau)\,p^c\right) - q\left(p^c\right) = 0, \end{aligned}$$

where the inequality takes into account that $q(\cdot)$ is non-decreasing.

- Therefore, $z((1+\tau)p^c) \geq 0$ and $z(p^p) = 0$. Since $z(\cdot)$ is non-increasing, this implies that $(1+\tau)p^c \leq p^p$. In words, levying the ad valorem tax on consumers leads to a lower cost on consumers than levying the same tax on producers. (In the same way, it can be shown that levying the ad valorem tax on consumers leads to a higher price for producers than levying the same tax on producers).

(d) Are there any special cases in which the collection method is irrelevant with an ad valorem tax? [*Hint:* Think about cases in which the tax introduces the same wedge on consumers and producers (inelasticity). Then prove your statement by using the above argument on excess demand functions.]

- If the supply function $q(\cdot)$ is strictly increasing, the argument can be strengthened to obtain the strict inequality: $(1+\tau)p^c < p^p$. On the other hand, when the supply is perfectly inelastic, i.e., $q(p) = \overline{q}$ =constant, then yield

$$x((1+\tau)p^c) = \overline{q} = x(p^p),$$

and therefore $p^p = (1+\tau)p^c$. Here both taxes result in the same cost to consumers. However, producers still bear a higher burden when the tax is levied directly on them:

$$(1-\tau)p^p = (1-\tau)(1+\tau)p^c < p^c.$$

these prices are depicted in the next figure, where $x(p)$ reflects the demand function with no taxes and $x((1-\tau)p)$ represents the demand function with the ad valorem tax. While the inelastic supply curve guarantees that sales are unaffected by the tax (remaining at $\overline{q}$ units), the price that the producer receives drops from p^p to $(1-\tau)p^p$. Therefore, the two taxes are still not fully equivalent.

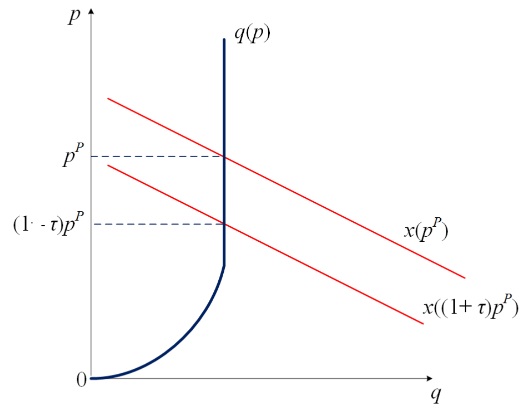

Figure 1. Introducing a tax.

- The intuition behind these results is simple: with a tax, there is always a wedge between the "consumer price"and the "producer price." Levying an ad valorem tax on the producer price, therefore, results in a higher tax burden (and a higher tax revenue) than levying the same percentage tax on consumers.

2. In our discussion of perfectly competitive markets, we considered that all firms produced a homogeneous good. However, our analysis can be easily extended to settings in which goods are heterogeneous. In particular, consider that every firm $i \in N$ faces a inverse demand function

$$p_i(q_i, q_{-i}) = \frac{\theta q_i^{\beta-1}}{\sum_{j=1}^{N} q_j^{\beta}}$$

where q_i denotes firm i's output, q_{-i} the output decisions of all other firms, i.e., $q_{-i} = (q_1, ..., q_{i-1}, q_{i+1}, ..., q_N)$, θ is a positive constant, and parameter $\beta \in (0, 1]$ captures the degree of substitutability. In addition, assume that every firm faces the same cost function $c(q_i) = F + cq_i$, where $F > 0$ denotes fixed costs and $c > 0$ represents marginal costs. Find the individual production level of every firm i, q_i^*, as a function of β. Interpret.

   - Every firm i's solves the following PMP

$$\max_{q_i} \frac{\theta q_i^{\beta-1}}{\sum_{j=1}^{N} q_j^{\beta}} q_i - (F + cq_i)$$

Taking first-order conditions with respect to q_i yields

$$\frac{\theta\left[\beta q_i^{\beta-1}\left(\sum_{j=1}^N q_j^\beta\right)-q_i^\beta\left(\beta q_i^{\beta-1}\right)\right]}{\left(\sum_{j=1}^N q_j^\beta\right)^2}-c=0$$

In a symmetric equilibrium, output levels satisfy $q_i^*=q^*$ for every firm $i\in N$, thus simplifying the above expression to

$$\frac{\theta\beta q^{2\beta-1}(N-1)}{N^2q^{2\beta}}-c=0$$

Solving for q^* yields the individual equilibrium output

$$q^*=\frac{\theta\beta(N-1)}{N^2c}$$

- *Comparative statics.* Differentiating q^* with respect to the substitutability parameter β we obtain
$$\frac{\partial q^*}{\partial\beta}=\frac{\theta(N-1)}{N^2c}>0$$
Hence, as goods become more differentiated (higher β), the equilibrium output level q^* rises. However, as more firms operate in this market, the increase in q^* becomes smaller since the derivative $\frac{\partial q^*}{\partial\beta}$ decreases in N, i.e.,
$$\frac{\partial\left(\frac{\partial q^*}{\partial\beta}\right)}{\partial N}=\frac{\theta N^2c-\theta(N-1)2Nc}{(N^2c)^2}<0.$$

3. Consider a polluting firm with profit function $\pi(q)=10q-q^2$ where q denotes units of the externality-generating activity (for instance, q can represent units of output if each unit generates one unit of pollution). Pollution damage to consumers is given by the convex damage function $d(q)=3q^2$. Let us analyze a context in which the regulator does not observe the firm's profit function, but observes the damage which additional pollution causes on consumers. In particular, the regulator estimates that marginal profits are
$$\frac{\partial\pi(q,a)}{\partial q}=10-2aq,$$
where the random parameter a takes two equally likely values, $a=1$ or $a=\frac{1}{2}$. (Note that in our above description we assume that the firm privately observes that the realization of parameter a is $a=1$, thus yielding a marginal profit function of $10-2q$.) We will first determine which is the best quota and emission fee that the regulator

can design given that he operates under incomplete information. Afterwards, we will evaluate the welfare that arises under each of these policy instruments, to determine which is better from a social point of view.

(a) *Unregulated equilibrium.* Find the equilibrium amount of pollution, q^E, if the firm is unregulated and no bargaining occurs between the affected consumers and the firm.

- In this setting, the firm maximizes its profits by solving

$$\max_q \pi(q)$$

Taking first-order conditions with respect to q, yields $\frac{\partial \pi(q^E)}{\partial q} \leq 0$. Since $\frac{\partial \pi(q)}{\partial q} = 10 - 2q$ by definition, then

$$\frac{\partial \pi(q^E)}{\partial q} = 10 - 2q^E \leq 0, \text{ with equality for } q^E > 0$$

Solving for q^E we obtain an equilibrium amount of pollution (in interior solutions) of $q^E = 5$ units.

(b) *Setting a quota.* In this incomplete information setting, determine which is the best quota x_q that a social planner can select in order to maximize the expected value of aggregate surplus.

- The firm must produce an output level exactly equal to the quota. The social planner determines the optimal quantity $\widehat{q}$ by choosing the value of q that maximizes the expected value of aggregate surplus (since the social planner does not know the precise realization of parameter a),

$$\max_q \; E_a[\pi(q,a)] - d(q)$$

And taking first order condition with respect to q, we obtain

$$E_a\left[\frac{\partial \pi(\widehat{q},a)}{\partial q}\right] - \frac{\partial d(\widehat{q})}{\partial q} \leq 0$$

We can now substitute the functional forms for the marginal damage for consumers, $\frac{\partial d(q)}{\partial q}$, and the expected marginal profits for the firm, $\frac{\partial \pi(q,a)}{\partial q}$, yielding

$$\frac{1}{2}(10 - 2 \cdot \widehat{q}) + \frac{1}{2}\left(10 - 2 \cdot \frac{1}{2} \cdot \widehat{q}\right) - 6\widehat{q} \leq 0.$$

which reduces to

$$5 - \widehat{q} + 5 - \frac{1}{2} \cdot \widehat{q} - 6\widehat{q} \leq 0, \text{ or } \quad \widehat{q} \geq \frac{4}{3}.$$

(c) *Setting an emission fee.* Find the best tax t^* that this social planner can set under the context of incomplete information described above.

- Given a tax t^*, the government predicts firm's expected best response function by maximizing its expected profits.

$$\max_{t} \quad E_a[\pi(q, a)] - tq$$

Taking first order condition with respect to t, yields

$$E_a\left[\frac{\partial \pi(q, a)}{\partial q}\right] - t = 0$$

and plugging our functional forms we obtain

$$5 - q + 5 - \frac{1}{2} \cdot q = t$$

which yields an output function $q(t) = \frac{20-2t}{3}$. Provided this expected output function, we can now find the optimal tax that the social planner imposes, anticipating the firm's expected best response function, as follows

$$\max_{t} \; E_a[\pi(q(t), a)] - d(q(t))$$

Taking first order conditions with respect to t, and applying the chain rule, yields

$$E\left[\frac{\partial \pi(q(t), a)}{\partial q} \cdot \frac{\partial q(t)}{\partial t}\right] = \frac{\partial d(q(t))}{\partial q} \cdot \frac{\partial q(t)}{\partial t}$$

where we use the chain rule. Intuitively, the regulator equals the marginal disutility of additional pollution to consumers (which he can perfectly assess), as represented in the right-hand side of the equality; and the expected marginal profits from additional pollution for the firm (which he cannot observe), represented in the left-hand side of the above expression.

- Since $q(t) = \frac{20-2t}{3}$ then the derivative $\frac{\partial q(t)}{\partial t} = -\frac{2}{3}$ is a constant, that can be taken out of the expectation operator. That is,

$$\frac{\partial q(t)}{\partial t} E\left[\frac{\partial \pi(q(t), a)}{\partial q}\right] = \frac{\partial d(q(t))}{\partial q} \cdot \frac{\partial q(t)}{\partial t}$$

Therefore, we can cancel out the $\frac{\partial q(t)}{\partial t}$ term on both sides of the equality, which yields

$$E\left[\frac{\partial \pi(q(t),a)}{\partial q}\right] = \frac{\partial d\left(q\left(t\right)\right)}{\partial q}.$$

Substituting the functional form of our marginal benefit and marginal profit functions, the above first-order condition becomes

$$\frac{1}{2}\left(10-2\cdot q\left(t\right)\right)+\frac{1}{2}\left(10-2\cdot\frac{1}{2}\cdot q\left(t\right)\right)=6q\left(t\right)$$

$$q\left(t\right)=\frac{4}{3}$$

Substituting $q\left(t\right)=\frac{20-2t}{3}$, we can finally find the optimal tax t^* that solves

$$\frac{20-2t^*}{3}=\frac{4}{3}, \quad \text{or} \quad t^*=8.$$

(d) *Policy comparison.* Compare the emission fee and the quota in terms of their associated deadweight loss. Under which conditions an uninformed regulator prefers to choose the emission fee?

- We need to compare the expected difference in losses in order to determine when a tax or a quota instrument is better. Figure 1 illustrates the welfare loss associated with tax t^*, which induces an externality level of $q\left(t^*\right)$.

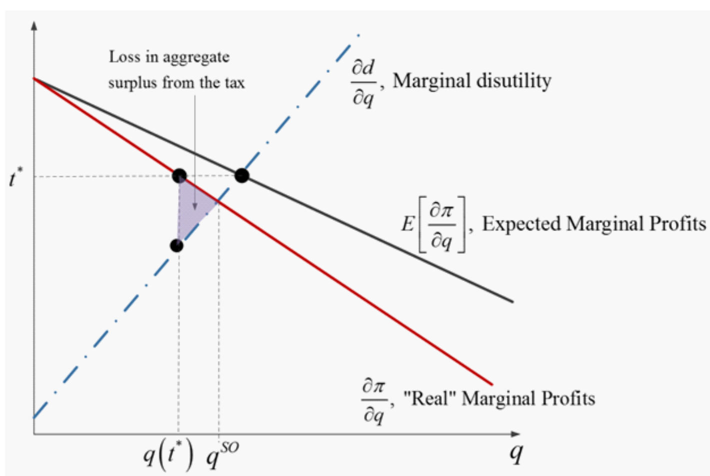

Figure 1. Welfare loss from a tax.

The figure considers that the regulator sets a tax based on the certain marginal disutility from the externality and the expected marginal profit. However, the realization of parameter a implies that the real and expected marginal profits do not coincide, thus giving rise to a welfare loss associated to a suboptimal tax due to the regulator's imprecise information.

- If the regulator, instead, imposes a quota, $\hat{q}$, figure 2 illustrates the associated welfare loss.

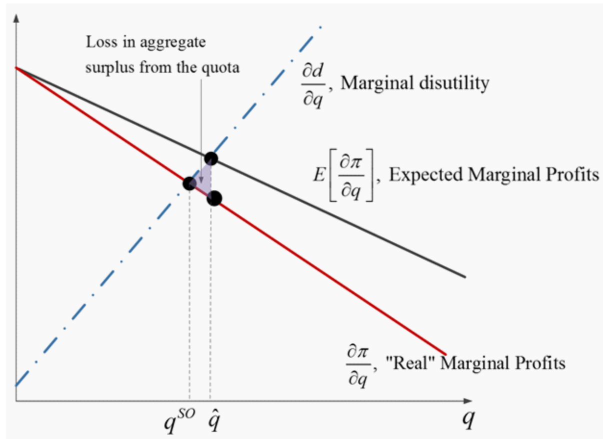

Figure 2. Welfare loss from a quota.

- *Welfare loss from the fee.* In order to compute the welfare loss from the tax, WL_t, we first need to find the socially optimal level of externality, q^{SO}, given the true $a = 1$. In particular, q^{SO} solves

$$\begin{aligned}\frac{\partial d(q^{SO})}{\partial q} &= \frac{\partial \pi(q^{SO})}{\partial q} \\ 6q^{SO} &= 10 - 2q^{SO} \\ q^{SO} &= \frac{5}{4}\end{aligned}$$

Moreover, given an emission fee, the firm maximizes profits. That is, it chooses the level of q that maximizes its profits (net of tax payments), as follows

$$\max_{q} \pi(q) - q \cdot t$$

The firm, hence, takes first order condition with repect to q, yielding

$$\frac{\partial \pi(q)}{\partial q} - t = 0$$

Since the firm knows its true marginal profit $\frac{\partial \pi(q)}{\partial q} = 10 - 2q$, the above expression becomes $10 - 2q - t = 0$, which yields an output function $q(t) =$

$5 - \frac{1}{2}t$. Given that $t^* = 8$, such fee induces an externality level of

$$q(t^*) = 5 - \frac{t^*}{2} = 1$$

Finally, we need to evaluate the marginal disutility function $\frac{\partial d(q)}{\partial q} = 6q$ at $q(t^*) = 1$, which yields $\frac{\partial d(q)}{\partial q} = 6q(t^*) = 6$. Hence, the WL_t is given by the area of the shaded triangle in figure 1,

$$\begin{aligned} WL_t &= \frac{1}{2}\left[q^{SO} - q(t^*)\right] \cdot [t^* - 6q(t^*)] \\ &= \frac{1}{2}\left[\frac{5}{4} - 1\right] \cdot [8 - 6] \\ &= \frac{1}{4} \end{aligned}$$

- *Welfare loss from the quota.* If, in contrast, the regulator uses a quota of $\widehat{q} = \frac{4}{3}$, then we first need to evaluate the real marginal profits of the quota, that is

  $$10 - 2q = 10 - 2 \times \frac{4}{3} = \frac{30}{3} - \frac{8}{3} = \frac{22}{3}$$

  Second, we need to evaluate the expected marginal profit, $\frac{1}{2}(10 - 2 \cdot q) + \frac{1}{2}\left(10 - 2 \cdot \frac{1}{2} \cdot q\right)$, at the quota $\widehat{q} = \frac{4}{3}$, i.e.,

  $$\frac{1}{2}\left(10 - 2 \cdot \frac{4}{3}\right) + \frac{1}{2}\left(10 - 2 \cdot \frac{1}{2} \cdot \frac{4}{3}\right) = 8.$$

  Therefore, the welfare loss from the quota is the area of the shaded triangle in figure 2. That is,

  $$\begin{aligned} WL_q &= \frac{1}{2}(\hat{q} - q^{SO})\left[8 - \frac{22}{3}\right] \\ &= \frac{1}{2}(\frac{4}{3} - \frac{5}{4})\left[\frac{24}{3} - \frac{22}{3}\right] \\ &= \frac{1}{36} \end{aligned}$$

- *Comparing welfare losses.* Comparing WL_t and WL_q, we obtain that

  $$WL_t = \frac{1}{4} > \frac{1}{36} = WL_q$$

  Hence, setting a quota is better than imposing an emission fee in this case.