Recitation #6 - Solution

1. Some indirect utility functions $v_i(p, w_i)$, such as those originating from a quasilinear preference relation, can be represented as a convex combination of individual i's wealth, w_i, as follows,

$$v_i(p, w_i) = a_i(p) + b(p)w_i$$

which are often referred as the Gorman form indirect utility function.

   (a) Show that if the indirect utility function can be expressed using the Gorman form, then all consumers exhibit parallel, straight wealth expansion paths at any price vector p. [*Hint:* Use Roy's identity].

   - To know the form of the wealth expansion paths, we need to know how the Walrasian demand $x_i(p, w)$ responds to changes in wealth levels. Since we have the consumer's indirect utility function $v_i(p, w_i)$, we can use Roy's identity in order to find his Walrasian demand $x_i(p, w)$

$$x_i(p, w) = -\frac{\frac{\partial v_i(p,w_i)}{\partial p}}{\frac{\partial v_i(p,w_i)}{\partial w}} = -\frac{a_i'(p) + b'(p)w_i}{b(p)}$$

We can next evaluate how the consumer's Walrasian demand $x_i(p, w)$ responds to changes in wealth, as follows

$$\frac{\partial x_i(p, w)}{\partial w} = -\frac{b'(p)b(p) - 0}{[b(p)]^2} = -\frac{b'(p)}{b(p)}$$

Hence, the slope of wealth expansion paths is given by $-\frac{b'(p)}{b(p)}$. In order to determine if wealth expansion paths are straight, we just have to check that their curvature does not change in w. Observing the above expression, we clearly see that wealth expansion paths have a slope which is constant in w. Additionally, note that the slope of the wealth expansion paths coincide across customers (no subscripts in the slope). Hence, the only difference in wealth expansion paths across consumers must be in the origin of this line, given by -$\frac{a_i'(p)}{b(p)}$, which varies across consumers.

   (b) Show also that, if the indirect utility function can be represented using the Gorman form, $v_i(p, w_i) = a_i(p) + b(p)w_i$, with the same $b(p)$ for all individuals, then the associated expenditure function, $e_i(p, u_i)$, can be expressed as

$$e_i(p, u_i) = c(p)u_i + d_i(p)$$

- By the Duality theorem, we have that the minimal expenditure needed to reach the utility level resulting from solving the UMP, $v_i(p, w_i)$, is

$$e_i\left(p, v_i(p, w_i)\right) = w_i$$

Solving for w_i in the Gorman-form indirect utility function $v_i(p, w_i) = a_i(p) + b(p)w_i$, we obtain

$$w_i = \frac{v_i(p, w_i) - a_i(p)}{b(p)}$$

Hence,

$$e_i\left(p, v_i(p, w_i)\right) = \frac{v_i(p, w_i) - a_i(p)}{b(p)}$$

And rearranging terms,

$$e_i\left(p, v_i(p, w_i)\right) = \frac{1}{b(p)} v_i(p, w_i) - \frac{a_i(p)}{b(p)}$$

or more compactly,

$$e_i\left(p, v_i(p, w_i)\right) = c(p) \cdot v_i(p, w_i) - d_i(p)$$

where we defined $c(p) \equiv \frac{1}{b(p)}$, which is constant for all i, and $d_i(p) \equiv \frac{a_i(p)}{b(p)}$, which is type-dependent. Therefore, we obtained that, if preferences admit Gorman-form indirect utility function with the same $b(p)$ for all individuals, then preferences admit expenditure functions of the form $e_i\left(p, v_i(p, w_i)\right) = c(p) \cdot v_i(p, w_i) - d_i(p)$.

1. Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce one type of output. The price of every unit of input is $w = 8$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem, and solve for its unconditional factor demand $z(8, p)$.

   - The firm chooses the units of input z to solve

   $$\max_{z \geq 0} p\sqrt{z} - 8z$$

   where the first term indicates total revenue, whereas the second reflects total costs. Taking first-order condition with respect to z, we obtain

$$p\frac{1}{2}z^{-1/2}-8\leq 0.$$

In the case of interior solutions, we can solve for z to find the unconditional factor demand

$$z(8,p)=\frac{p^2}{256}.$$

Hence, total output is $q=\sqrt{\frac{p^2}{256}}=\frac{p}{16}$ units.

(b) Evaluate the profit function at the unconditional factor demand $z(8,p)$. Test for convexity of the profit function in output price p.

- Inserting $z(8,p)=\frac{p^2}{256}$ into the firm's objective function, we obtain

$$\pi(p)=p\sqrt{z(8,p)}-8z(8,p)=\frac{1}{32}p\,(2p-p)=\frac{p^2}{32},$$

which is convex in output price p.

(c) Let us now illustrate convexity in output prices by using an alternative approach: (1) evaluate the profit function you found in part (b) at prices $p=6$, and at $p=12$. Then, find their convex combination $\alpha\pi(6)+(1-\alpha)\,\pi(12)$ where $\alpha\in[0,1]$; (2) evaluate the profit function at the convex combination of the above output prices, that is, $\pi(\alpha 6+(1-\alpha)\,12)$. Last, show that the profit function you found in step (1) lies weakly above that found in step (2) for all values of α, that is,

$$\alpha\pi(6)+(1-\alpha)\,\pi(12)\geq\pi(\alpha 6+(1-\alpha)\,12).$$

- Evaluating the output function at those two output prices, we obtain $\pi(6)=\frac{9}{8}$ and $\pi(12)=\frac{9}{2}$. Hence, their convex combination is

$$\alpha\pi(6)+(1-\alpha)\,\pi(12)=\alpha\frac{9}{8}+(1-\alpha)\,\frac{9}{2}=\frac{9}{8}(4-3\alpha).$$

If, instead, we evaluate the profit function at an output price $p=\alpha 6+(1-\alpha)\,12$, we obtain

$$\pi\left(\alpha 6+(1-\alpha)\,12\right)=\frac{9}{8}(4-4\alpha+\alpha^2)$$

Substracting $[\alpha\pi(6)+(1-\alpha)\,\pi(12)]-\pi(\alpha 6+(1-\alpha)\,12)$, we find

$$\frac{9}{8}(4-3\alpha)-\frac{9}{8}(4-4\alpha+\alpha^2)=\frac{9}{8}\alpha\,(1-\alpha)>0.$$

which is positive since $\alpha \in [0, 1]$.