EconS 501 Midterm #1 - October 12th, 2020

Show all your work clearly and make sure you justify all your answers.
NAME_________________________

1. For each of the following utility functions, show if they are homothetic.

   (a) Cobb-Douglas utility function $u(x,y) = Ax^{\alpha}y^{\beta}$, where $A, \alpha, \beta > 0$.

   - The marginal rate of substitution is

   $$MRS_{x,y} = \frac{A\alpha x^{\alpha-1}y^{\beta}}{A\beta x^{\alpha}y^{\beta-1}} = \frac{\alpha y}{\beta x}.$$

   Hence, for a given $\frac{y}{x}$ ratio, the $MRS_{x,y}$ is constant. Graphically, if we depict a ray from the origin with slope $\frac{y}{x}$, the slope of all indifference curves crossed by this ray is the same. Therefore, homotheticity holds.

   (b) Goods are regarded as substitutes $u(x,y) = ax + by$, where $a, b > 0$.

   - The marginal rate of substitution is

   $$MRS_{x,y} = \frac{a}{b}.$$

   Hence, for a given $\frac{y}{x}$ ratio, the $MRS_{x,y}$ is constant. Graphically, if we depict a ray from the origin with slope $\frac{y}{x}$, the slope of all indifference curves crossed by this ray is the same. Therefore, homotheticity holds.

   (c) Goods are regarded as complements $u(x,y) = \min\{ax, by\}$, where $a, b > 0$.

   - In this setting, the $MRS_{x,y}$ is undefined. Nonetheless, if we depict a ray from the origin with slope $\frac{y}{x}$, the slope of all indifference curves crossed by this ray is the same. Therefore, homotheticity holds.

   (d) Stone-Geary utility function $u(x,y) = A\left(x - \overline{x}\right)^{\alpha}\left(y - \overline{y}\right)^{\beta}$, where $A, \alpha, \beta \in \mathbb{R}$ and $\overline{x}, \overline{y} > 0$.

   - The marginal rate of substitution is

   $$MRS_{x,y} = \frac{A\alpha\left(x - \overline{x}\right)^{\alpha-1}\left(y - \overline{y}\right)^{\beta}}{A\beta\left(x - \overline{x}\right)^{\alpha}\left(y - \overline{y}\right)^{\beta-1}} = \frac{\alpha\left(y - \overline{y}\right)}{\beta\left(x - \overline{x}\right)}.$$

   In this setting, the consumer will always consume at least $(\overline{x}, \overline{y})$ irrespective of her budgets or the prices. The $MRS_{x,y}$ also depends on $(\overline{x}, \overline{y})$ given any consumption bundle (x, y). That is the $MRS_{x,y}$ is increasing when both x and y increase. Therefore,homotheticity does not hold.

   (e) Highly non-linear utility function $u(x,y) = Ax^{\alpha}y + Byx^{\beta}$, where $A, B, \alpha, \beta > 0$.

   - The marginal rate of substitution is

   $$MRS_{x,y} = \frac{A\alpha x^{\alpha-1}y + B\beta x^{\beta-1}y}{Ax^{\alpha} + Bx^{\beta}} = f\left(x\right)\frac{y}{x}.$$

   where $f(x) = \frac{A\alpha x^{\alpha} + B\beta x^{\beta}}{Ax^{\alpha} + Bx^{\beta}}$. In this setting, the $MRS_{x,y}$ is not constant given any $\frac{y}{x}$ because $f(x)$ is a non-linear function of x. Therefore,homotheticity does not hold.

2. Consider a firm with the following production function $q = f(w,k) = \frac{\theta}{1+w^{-\delta}k^{-\varepsilon}}$, where $\theta, \delta, \varepsilon > 0$ and w and k denote water and minerals, respectively. The firm has a new technology that allows it to transform all remaining outputs back to the initial amounts of inputs. In addition, the producer can eliminate additional inputs at a cost equal to $\$X$.

   (a) Define the production set and discuss what properties of the production set are not satisfied.

   - The production set is

$$Y = \{(w,k) : q \leq f(w,k) = \frac{\theta}{1+w^{-\delta}k^{-\varepsilon}}\}$$

   The production set does not satisfy: free disposal and irreversibility.

   (b) Identify the sufficient conditions that guarantee that the $MRTS_{l,k}$ is diminishing.

   - First, identify the marginal product (MP_w and MP_k) of water and minerals

$$\begin{aligned}\frac{\partial f(w,k)}{\partial w} &= \frac{\theta\delta w^{\delta-1}k^{\varepsilon}}{(1+w^{\delta}k^{\varepsilon})^2} > 0\\ \frac{\partial f(w,k)}{\partial k} &= \frac{\theta\varepsilon w^{\delta}k^{\varepsilon-1}}{(1+w^{\delta}k^{\varepsilon})^2} > 0\end{aligned}$$

   - Second, find the MP_{ww} and MP_{kk}

$$\begin{aligned}\frac{\partial MP_w}{\partial w} &= -\frac{\delta w^{\delta-2}k^{\varepsilon}(1-\delta+w^{\delta}k^{\varepsilon}(1+\delta))\theta}{(1+w^{\delta}k^{\varepsilon})^3} < 0\\ \frac{\partial MP_k}{\partial k} &= -\frac{\varepsilon w^{\delta}k^{-2+\varepsilon}(1-\varepsilon+w^{\delta}k^{\varepsilon}(1+\varepsilon))\theta}{(1+w^{\delta}k^{\varepsilon})^3} < 0\end{aligned}$$

   - Finally, we need to check that $MP_{wk} = MP_{kw}$ and under which conditions they are positive

$$\begin{aligned}\frac{\partial MP_w}{\partial k} &= -\frac{\delta\varepsilon w^{-1+\delta}k^{-1+\varepsilon}(-1+w^{\delta}k^{\varepsilon})\theta}{(1+w^{\delta}k^{\varepsilon})^3} > 0\\ \frac{\partial MP_k}{\partial w} &= -\frac{\delta\varepsilon w^{-1+\delta}k^{-1+\varepsilon}(-1+w^{\delta}k^{\varepsilon})\theta}{(1+w^{\delta}k^{\varepsilon})^3} > 0\end{aligned}$$

   which is positive if and only if $w^{\delta}k^{\varepsilon} < 1$ or $k < w^{\frac{-\delta}{\varepsilon}}$.

   (c) Find the product-elasticities of the two inputs and comment the type of returns to scale that the function represents. Use a numerical example considering parameters $\delta = \varepsilon = \frac{3}{4}$ and discuss your results.

   - *Product elasticity.* The product-elasticity of the input j, $\varepsilon_{q,j}$ where $j = \{w,k\}$, is defined as the percentage change in output $q = f(z)$ with respect to a percentage change in the amount used of input z_i. We can calculate it as:

$$\varepsilon_{q,j} \equiv \frac{\partial f(w,k)}{\partial j}\frac{j}{f(w,k)}$$

For the case of this particular functional form, we have that the product elasticity of input 1 is

$$\begin{aligned}\varepsilon_{q,w} &= \frac{\partial f(w,k)}{\partial w}\frac{w}{f(w,k)} = -\theta(1+w^{-\delta}k^{-\varepsilon})^{-2}(-\delta)(w^{-\delta-1}k^{-\varepsilon})\frac{w}{\theta(1+w^{-\delta}k^{-\varepsilon})^{-1}} \\ &= \delta(1+w^{-\delta}k^{-\varepsilon})^{-1}w^{-\delta}k^{-\varepsilon}\end{aligned}$$

and the product elasticity of input 2 is

$$\begin{aligned}\varepsilon_{q,z_2} &= \frac{\partial f(w,k)}{\partial k}\frac{k}{f(w,k)} = -\theta(1+w^{-\delta}k^{-\varepsilon})^{-2}(-\varepsilon)(w^{-\delta}k^{-\varepsilon-1})\frac{k}{\theta(1+w^{-\delta}k^{-\varepsilon})^{-1}} \\ &= \varepsilon(1+w^{-\delta}k^{-\varepsilon})^{-1}w^{-\delta}k^{-\varepsilon}\end{aligned}$$

- *Scale elasticity.* The scale elasticity can help us find this production function's returns to scale. In particular, define elasticity of scale, $\varepsilon_{q,t}$, as the percentage change in total output as a consequence of a percentage change in *all* inputs by a common factor t. That is,

$$\varepsilon_{q,t} \equiv \lim_{t\to 1}\frac{\partial f(tz)}{\partial t}\frac{t}{f(tz)}$$

  Alternatively, the scale elasticity can be calculated as the sum of the product-elasticities for all inputs in the production process:

$$\varepsilon_{q,t} = \sum_{i=1}^{n}\varepsilon_{q,j}$$

  which in this case is

$$\begin{aligned}\varepsilon_{q,t} &= \varepsilon_{q,w}+\varepsilon_{q,k} = \delta(1+w^{-\delta}k^{-\varepsilon})^{-1}w^{-\delta}k^{-\varepsilon}+\varepsilon(1+w^{-\delta}k^{-\varepsilon})^{-1}w^{-\delta}k^{-\varepsilon} \\ &= (\delta+\varepsilon)(1+w^{-\delta}k^{-\varepsilon})^{-1}w^{-\delta}k^{-\varepsilon}\end{aligned}$$

- *Returns to scale.* This function does not represent global, but local, returns to scale. That is, the type of returns to scale depends on the production level (or the amount of inputs used). We can use the scale elasticity to determine for which values of inputs z_1 and z_2 the production function exhibits constant, increasing or decreasing returns to scale, as follows.
  - For values of w and k for which $\varepsilon_{q,t} = 1$,

$$(\delta+\varepsilon)(1+w^{-\delta}k^{-\varepsilon})^{-1}w^{-\delta}k^{-\varepsilon} = 1$$

    the production function exhibits *constant* returns to scale.
  - For values of w and k for which $\varepsilon_{q,t} > 1$,

$$(\delta+\varepsilon)(1+w^{-\delta}k^{-\varepsilon})^{-1}w^{-\delta}k^{-\varepsilon} > 1$$

    the production function has *increasing* returns to scale.

- Finally, for values of w and k for which $\varepsilon_{q,t} < 1$,

$$(\delta + \varepsilon)(1 + w^{-\delta}k^{-\varepsilon})^{-1}w^{-\delta}k^{-\varepsilon} < 1$$

  the production function exhibits *decreasing* returns to scale.

- *Numerical example.* For instance, for parameters $\delta = \varepsilon = \frac{3}{4}$, we obtain

$$\left(\frac{3}{4} + \frac{3}{4}\right)\left(1 + w^{-\frac{3}{4}}k^{-\frac{3}{4}}\right) w^{-\frac{3}{4}}k^{-\frac{3}{4}} \equiv \varepsilon_{q,t}(w,k)$$

  which we denote as $\varepsilon_{q,t}(w,k)$ for compactness.

  - Setting $\varepsilon_{q,t}(w,k) = 1$ and solving for k, yields $k = \frac{0.59}{w}$, which describes the level set of (w,k) −pairs for which the scale elasticity is equal to 1 (implying that the production function exhibits constant returns to scale).
  - Similarly, setting $\varepsilon_{q,t}(w,k) > 1$, e.g., $\varepsilon_{q,t}(w,k) = \frac{3}{2}$ and solving for k, yields $k = \frac{0.51}{w}$, which represents the level set of (w,k) −pairs for which the scale elasticity is $\frac{3}{2}$ (implying that the production function exhibits increasing returns to scale). Intuitively, when the firm uses few units of inputs, the scale elasticity is larger than 1, indicating that a given increase in both w and k entails a more-than-proportional increase in output. However, when the firm already uses large amounts of both inputs, a further increase in w and k yields a proportional increase in output; (If such amount of inputs is sufficiently large, then $\varepsilon_{q,t}(w,k) < 1$, suggesting that output increases less than proportionally.)

3. Assume that $L = 2$ and consider a *local* indirect utility function defined in some neighborhood of price-wealth pair $(\widetilde{p}, \widetilde{w})$ by

$$v(p,w) = \left[\frac{w}{p_2} + \frac{1}{b}\left(a\frac{p_1}{p_2} + \frac{a}{b} + c\right)\right] \exp\left(-\frac{bp_1}{p_2}\right)$$

   (a) Verify that the local demand function for good 1 is

$$x_1(p,w) = a\frac{p_1}{p_2} + b\frac{w}{p_2} + c$$

   1. We can use Roy's identity to verify $x_1(p,w)$. That is

$$x_1(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}}$$

   where

$$\frac{\partial v(p,w)}{\partial p_1} = -\frac{(ap_1 + bw + cp_2)\exp\left(-\frac{bp_1}{p_2}\right)}{p_2^2}$$

$$\frac{\partial v(p,w)}{\partial w} = \frac{\exp\left(-\frac{bp_1}{p_2}\right)}{p_2}$$

Hence,

$$x_1(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\left(-\frac{(ap_1+bw+cp_2)\exp\left(-\frac{bp_1}{p_2}\right)}{p_2}\right)}{\frac{\exp\left(-\frac{bp_1}{p_2}\right)}{p_2}} = a\frac{p_1}{p_2} + b\frac{w}{p_2} + c$$

(b) Verify that the local expenditure function is

$$e(p,u) = p_2 u \exp\left(\frac{bp_1}{p_2}\right) - \frac{1}{b}(ap_1 + \frac{a}{b}p_2 + cp_2)$$

In order to find the expenditure function we can use

$$\begin{aligned} v(p, e(p,u)) &= u \\ \left[\frac{p_2 u \exp\left(\frac{bp_1}{p_2}\right) - \frac{1}{b}(ap_1 + \frac{a}{b}p_2 + cp_2)}{p_2} + \frac{1}{b}(a\frac{p_1}{p_2} + \frac{a}{b} + c)\right]\exp\left(-\frac{bp_1}{p_2}\right) &= u \end{aligned}$$

which holds.

(c) Verify that the Hicksian demand function for good 1 is

$$h_1(p,u) = ub\exp\left(\frac{bp_1}{p_2}\right) - \frac{a}{b}$$

from Shepard's lemma we know that

$$\begin{aligned} \frac{\partial e(p,u)}{\partial p_1} &= h_1(p,u) \\ \frac{\partial e(p,u)}{\partial p_1} &= ub\exp\left(\frac{bp_1}{p_2}\right) - \frac{a}{b} \end{aligned}$$

Hence, $h_1(p,u)$ is correct.

4. Assume that you have friend, Leo, who is now retired and lives on a fixed income $w > 0$ which does not adjust to inflation. His expenditure function is

$$e(p_1, p_2, u) = (p_1 + p_2)u,$$

where $p_1, p_2 > 0$ denote initial prices. Suppose that prices of goods 1 and 2 increase to p_1' and p_2', respectively.

(a) You want to give him a monetary gift so that he will not be affected by the above price increase. How much money should you give him? That is, find his compensating variation (CV).

- From duality, we know that $e(p_1, p_2, u) = w$, which helps us rewrite the above equation as $w = (p_1 + p_2)u$. Therefore, at the initial price vector, the maximum utility the retiree can obtain is $u = \frac{w}{p_1+p_2}$. In order to ensure that he is not worse off after the price increase, we need to give him an amount of money that covers the difference between the new expense and the old expense, that is,

$$\begin{aligned} CV &= w' - w \\ &= e(p_1', p_2', u) - w \\ &= \underbrace{(p_1' + p_2')u}_{e(p_1', p_2', u)} - w \\ &= (p_1' + p_2') \underbrace{\frac{w}{p_1 + p_2}}_{u} - w \\ &= w\left(\frac{p_1' + p_2'}{p_1 + p_2} - 1\right) \end{aligned}$$

(b) Now find his equivalent variation (EV) from the price change, i.e., the change in income needed at initial prices p_1 and p_2 that would have the same effect on utility as would the change in prices, p_1' and p_2'.

- The equivalent variation is

$$\begin{aligned} EV &= e(p_1, p_2, u') - e(p_1', p_2', u') \\ &= (p_1 + p_2)u' - (p_1' + p_2')u' \\ &= (p_1 + p_2) \underbrace{\frac{w}{p_1' + p_2'}}_{u'} - (p_1' + p_2') \underbrace{\frac{w}{p_1' + p_2'}}_{u'} \\ &= w\left(\frac{p_1 + p_2}{p_1' + p_2'} - 1\right) \end{aligned}$$

(c) Which is larger in this case, CV or EV?

- In this case, since $p_1' + p_2' > p_1 + p_2$, we obtain that the compensating variation is positive $CV > 0$, but the equivalent variation is negative $EV < 0$, entailing that $CV > 0 > EV$.

(d) Find his Walrasian demand for each good.

- *Good 1.* From duality, we know that $e(p_1, p_2, u) = w$, which yields $w = (p_1 + p_2)u$. Solving for u, we find the indirect utility function

$$v(p_1, p_2) = (p_1 + p_2)^{-1} w$$

Then, we can insert this indirect utility function into Roy's identity, as follows, which yields the Walrasian demand for good 1:

$$x_1(p_1, p_2, w) = -\frac{\frac{\partial v(p_1, p_2)}{\partial p_1}}{\frac{\partial v(p_1, p_2)}{\partial w}} = \frac{w(p_1 + p_2)^{-2}}{(p_1 + p_2)^{-1}} = w(p_1 + p_2)^{-1}.$$

- *Good 2.* Following a similar approach, we can find the Walrasian demand for good 2:

$$x_2(p_1, p_2, w) = -\frac{\frac{\partial v(p_1, p_2)}{\partial p_2}}{\frac{\partial v(p_1, p_2)}{\partial w}} = \frac{w(p_1 + p_2)^{-2}}{(p_1 + p_2)^{-1}} = w(p_1 + p_2)^{-1}$$

implying that the consumer purchases the same amount of both goods, $x_1(p_1, p_2, w) = x_2(p_1, p_2, w)$.

(e) Find his utility function. What is this type of utility function called?

- In part (d), we found that the consumer in this exercise, Leo, purchases the same amount of both goods, $x_1(p_1, p_2, w) = x_2(p_1, p_2, w)$, regardless of the price vector and income he faces. This only occurs when the consumer regards both goods as complements, exhibiting a Leontief utility function $u(x_1, x_2) = A \min\{x_1, x_2\}$, where $A > 0$. As a remark, note that we do not say that his utility function is the general expression of the Leontief utility function $u(x_1, x_2) = A \min\{ax_1, bx_2\}$, where $A, a, b > 0$ since in such a setting the consumer could purchase different amounts of each good; as long as he keeps the proportion of goods he consumes constant.