Homework #5 (10/05/2020)

1. Consider a society where every individual i has the following (discontinuous) Walrasian demand function

$$x_i(p) = \begin{cases} \frac{w}{4p} \text{ if } p > k, \\ \frac{w}{2p} \text{ if } p < k, \text{ and} \\ \frac{w}{4p} \text{ or } \frac{w}{2p} \text{ if } p = k \end{cases}$$

where k is a positive constant, while $w > 0$ denotes income.

(a) Assume that there are only two individuals in this society, 1 and 2. If they both have the same income $w > 0$, find their average demand, i.e., $\frac{x_1+x_2}{2}$. Then show that the average demand $\frac{x_1+x_2}{2}$ takes three possible values at $p = k$.

- There are three situations we need to consider:
  1. If the demand of both consumers 1 and 2 is $x_i(p) = \frac{w}{4p}$, which can be expressed as $\frac{w}{4k}$ since $p = k$, then average demand is

$$\frac{x_1 + x_2}{2} = \frac{\frac{w}{4k} + \frac{w}{4k}}{2} = \frac{w}{4k}$$

  2. If both consumer 1's and 2's demand is $x_i(p) = \frac{w}{2p}$, which can be expressed as $\frac{w}{2k}$ since $p = k$, average demand becomes

$$\frac{x_1 + x_2}{2} = \frac{\frac{w}{2k} + \frac{w}{2k}}{2} = \frac{w}{2k}$$

  3. Finally, if one of the consumers demands $x_i(p) = \frac{w}{4p}$ while the other demands $x_j(p) = \frac{w}{2p}$ where $j \neq i$, average demand in this case is

$$\frac{x_i + x_j}{2} = \frac{\frac{w}{4k} + \frac{w}{2k}}{2} = \frac{3w}{8k}$$

(b) Now consider a society with an infinite number of individuals, all with the above Walrasian demand $x_i(p)$ and the same income $w > 0$. Show that now the average demand at $p = k$ now takes all the values between $\frac{w}{4p}$ and $\frac{w}{2p}$.

- If the number of individuals is now generically N, we can still reproduce the analysis in part (b) by noticing that, at $p = k$, N_1 indviduals demand $x_i(p) = \frac{w}{4p} = \frac{w}{4k}$ while the remaining N_2 individuals demand $x_i(p) = \frac{w}{2p} = \frac{w}{2k}$, where $N_1 + N_2 = N$. As a consequence, average demand becomes

$$\frac{N_1 \frac{w}{4k} + (N - N_1) \frac{w}{2k}}{N}$$

which simplifies to

$$\frac{w}{2k}\left(1-\frac{N_1}{2N}\right)$$

Hence, average demand must belong to the set $\frac{w}{2k}\left(1-\frac{N_1}{2N}\right)$ for all $N_1 = 1, 2, ..., N$. When $N \to \infty$, this set becomes dense in the segment $\left[\frac{w}{4k}, \frac{w}{2k}\right]$. As a consequence, aggregate demand "fills" the discontinuity segment of every individual i's Walrasian demand.

2. Suppose that a set of I consumers all have homothetic preferences represented by utility functions that satisfy homogeneity of degree one. Consider now the social welfare function

$$W(u_1, u_2, ..., u_I) = \sum_{i=1}^{I} \alpha_i \ln u_i \quad \text{with } \alpha_i > 0 \text{ and } \sum_{i=1}^{I} \alpha_i = 1$$

Show that the optimal wealth distribution rule (emerging from the social welfare maximization problem) is

$$w_i(p, w) = \alpha_i w$$

That is, the optimal wealth distribution rule assigns a constant proportion of wealth to every individual, irrespective of the price level.

- The social welfare maximization problem is now written as selecting a distribution of wealth among the I individuals $(w_1, w_2, ..., w_I)$ that solves

$$\max_{(w_1,...,w_I)} \quad W = \sum_i \alpha_i \ln v_i(p, w_i)$$

$$\text{subject to} \quad \sum_i w_i \leq w$$

Hence, the Lagrangian associated to this maximization problem is

$$W - \lambda(w_i + w)$$

Taking first order conditions with respect to w_i, yields

$$\underbrace{\left(\frac{\partial W}{\partial v_i}\right)}_{\text{Term 1}}\underbrace{\left(\frac{\partial v_i}{\partial w_i}\right)}_{\text{Term 2}} = \lambda$$

for every individual i, where all derivatives are evaluated at a solution $(w_1, ..., w_I)$. Let us separately analyze each of the two terms.

– First, using the definition of the social welfare function $W(\cdot) = \sum_i \alpha_i \ln v_i(p, w_i)$, we can rewrite Term 1 as follows

$$\frac{\partial W}{\partial v_i} = \frac{\alpha_i}{v_i(p, w_i)}.$$

– Regarding Term 2, i.e., $\frac{\partial v_i}{\partial w_i}$, we can use Euler's theorem: in particular, since $v_i(p, w_i)$ is homogeneous of degree one in w_i,[1]

$$v_i(p, w_i) = w_i \frac{\partial v_i(p, w_i)}{\partial w_i},$$

and solving for $\frac{\partial v_i(p,w_i)}{\partial w_i}$ yields

$$\frac{\partial v_i(p, w_i)}{\partial w_i} = \frac{v_i(p, w_i)}{w_i}.$$

Hence, replacing Terms 1 and 2 in the above first-order condition, we obtain

$$\underbrace{\frac{\alpha_i}{v_i(p, w_i)}}_{\text{Term 1}} \underbrace{\frac{v_i(p, w_i)}{w_i}}_{\text{Term 2}} = \lambda$$

which simplifies to $\frac{\alpha_i}{w_i} = \lambda$ for every i. Thus, solving for w_i, we obtain $w_i = \alpha_i \frac{1}{\lambda}$. Since $\sum_i \alpha_i = 1$ and $\sum_i w_i = w$, then $\sum_i w_i = w = \frac{1}{\lambda}$. Hence, we can rewrite $w_i = \alpha_i \frac{1}{\lambda}$ as $w_i = \alpha_i w$. That is, the welfare-maximizing distribution of wealth $w_i(p, w)$ prescribes that each individual should receive a type-dependent share of the aggregate wealth $w_i(p, w) = \alpha_i w$.

3. Suppose that a firm owns two plants, each producing the same good. Every plant j's average cost is given by

$$AC_j(q_j) = \alpha + \beta_j q_j \quad \text{for} \quad q_j \geq 0, \text{ where } j = \{1, 2\}$$

where coefficient β_j may differ from plant to plant, i.e., if $\beta_1 > \beta_2$ plant 2 is more efficient than plant 1 since its average costs increase less rapidly in output. Assume that you are asked to determine the cost-minimizing distribution of aggregate output

[1] Recall that Euler's theorem states that, if a function $f(x)$ is homogeneous of degree k, then

$$\frac{\partial f(x)}{\partial x} x = t^k \cdot f(x)$$

where t is the proportionality factor. Hence, if function $f(x)$ is homogeneous of degree zero (as the indirect utility function), $k = 0$, and Euler's theorem simplifies to $\frac{\partial f(x)}{\partial x} x = f(x)$.

$q = q_1 + q_2$, among the two plants (i.e., for a given aggregate output q, how much q_1 to produce in plant 1 and how much q_2 to produce in plant 2.) For simplicity, consider that aggregate output q satisfies $q < \frac{\alpha}{\max_j |\beta_j|}$. (You will be using this condition in part b.)

(a) If $\beta_j > 0$ for every plant j, how should output be located among the two plants?

- The cost-minimization problem in which we find the optimal combination of output q_1 and q_2 that minimizes the total cost of production across plants is

$$\min_{q_1, q_2} \; TC_1(q_1) + TC_2(q_2)$$

$$\text{subject to } \; q_1 + q_2 = q$$

  or equivalently, the profit maximization problem in which firms choose the optimal combination of output q_1 and q_2 that maximizes the total profits across all plants is

$$\max_{q_1, q_2} \; \underbrace{pq_1 - TC_1(q_1)}_{\pi_1} + \underbrace{pq_2 - TC_2(q_2)}_{\pi_2}$$

$$\text{subject to } \; q_1 + q_2 = q$$

- If the average cost is $AC_j(q_j) = \alpha + \beta_j q_j$ then the total cost is $TC_j(q_j) = (\alpha + \beta_j q_j) q_j$. Thus, we can rewrite the above PMP as:

$$\max_{q_1, q_2} \; pq_1 - (\alpha + \beta_1 q_1) q_1 + pq_2 - (\alpha + \beta_2 q_2) q_2$$

$$\text{subject to } \; q_1 + q_2 = q$$

  Taking first order conditions with respect to q_1 and q_2 yields

$$\frac{\partial (\pi_1 + \pi_2)}{\partial q_1} = p - \alpha - 2\beta_1 q_1 = \lambda$$

$$\frac{\partial (\pi_1 + \pi_2)}{\partial q_2} = p - \alpha - 2\beta_2 q_2 = \lambda$$

$$\frac{\partial (\pi_1 + \pi_2)}{\partial \lambda} = q_1 + q_2 = q$$

  Using the first two order conditions, we obtain

$$p - \alpha - 2\beta_1 q_1 = p - \alpha - 2\beta_2 q_2$$

and rearranging, $q_2 = \frac{\beta_1}{\beta_2} q_1$. Replacing this expression into the constraint $q_1 + q_2 = q$ yields

$$q_1 + \underbrace{\frac{\beta_1}{\beta_2} q_1}_{q_2} = q$$

and solving for q_1 entails the cost-minimizing production in plant 1,

$$q_1 \left(1 + \frac{\beta_1}{\beta_2}\right) = q, \quad \text{thus} \quad q_1 = \frac{\beta_2}{\beta_1 + \beta_2} q,$$

and operating similarly for q_2, we find

$$q_2 = \frac{\beta_1}{\beta_1 + \beta_2} q$$

- *Extension:* Note that, generally for J plants, the average cost of plant j is $AC_j(q_j) = \alpha + \beta_j q_j$ implying that the total cost must be $TC_j(q_j) = (\alpha + \beta_j q_j) q_j$. Therefore, plant j's marginal cost is $MC_j(q_j) = \alpha + 2\beta_j q_j$. Since $\beta_j > 0$ for every j, the first order necessary and sufficient conditions for cost minimization are: (1) that firms' marginal costs coincide (otherwise, we would still have incentives to distribute a larger production to those firms with the lowest marginal cost)

$$MC_j(q_j) = MC_{j'}(q_{j'}) \text{ for any two plants } j \text{ and } j'$$

and; (2) that the aggregate output constraint holds

$$q_1 + q_2 + ...q_J = q.$$

From these conditions we obtain

$$q_j = \frac{\frac{q}{\beta_j}}{\sum_h \frac{1}{\beta_h}}.$$

which coincides with our results for $N = 2$ plants,

$$q_1 = \frac{\frac{q}{\beta_1}}{\frac{1}{\beta_1} + \frac{1}{\beta_2}} = \frac{\beta_2}{\beta_1 + \beta_2} q.$$

Figure 2 depicts the average and marginal cost curves for two plants satisfying $\beta_2 > \beta_1$. In particular, the firm manager chooses, for a given aggregate output $q = q_1 + q_2$, the individual output levels q_1 and q_2 that equate the marginal costs across both plants (see vertical axis).

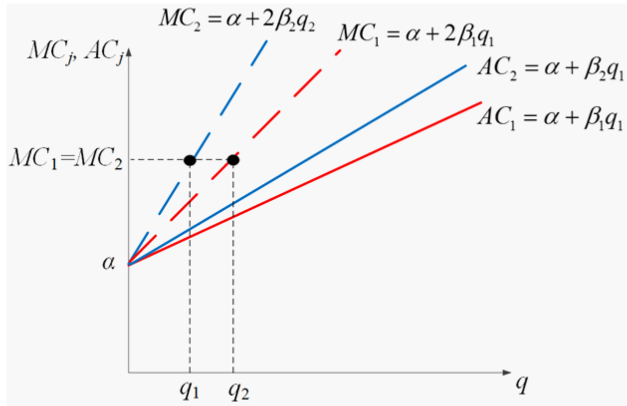

Figure 2. $\beta_j > 0$ for every firm.

(b) If $\beta_j < 0$ for every plant j, how should output be located among the two plants?

- First, note that $\beta_j < 0$ implies that the average cost $AC_j(q_j) = \alpha + \beta_j q_j$ is decreasing in output. Hence, it is cost-minimizing to concentrate all production on the plant with the smallest $\beta_j < 0$ (the most negative β_j) because average costs (and total costs) are minimized by doing so.
- Figure 3 depicts a firm in which both plants exhibit decreasing average costs, but $\beta_2 < \beta_1 < 0$, implying that it is beneficial for the firm to concentrate all output in plant 2. In addition, note that the average cost in plant 1 is positive for all q_1 as long as $\alpha - \beta_1 q_1 > 0$, or $q_1 < \frac{\alpha}{\beta_1}$, where $\frac{\alpha}{\beta_1}$ represents the horizontal intercept of AC_1 in the figure. Similarly for firm 2, where $AC_2 > 0$ for all q_2 as long as $q_2 < \frac{\alpha}{\beta_2}$, where $\frac{\alpha}{\beta_2}$ represents the horizontal intercept of AC_2. Hence, the original condition $q < \frac{\alpha}{\max_j |\beta_j|}$ is equivalent to $q < \min_j \frac{\alpha}{|\beta_j|}$, graphically implying that the aggregate output q lies to the left-hand side to

the smallest horizontal intercept.

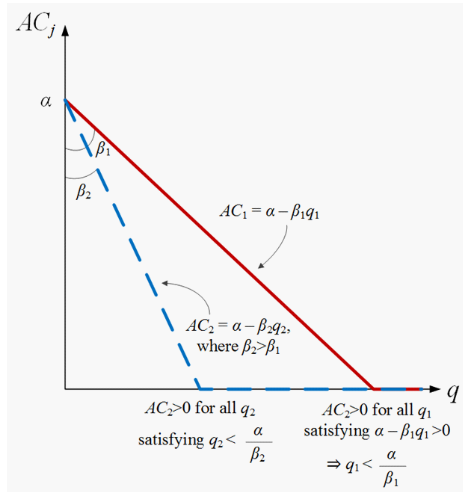

Figure 3. $\beta_j < 0$ for every firm.

(c) If $\beta_j > 0$ for some plants and $\beta_i < 0$ for others?

- Similarly as in part (b), the firm now faces some plants with increasing average costs (those with $\beta_j > 0$) and some plants with decreasing average costs (those with $\beta_j < 0$). Hence, it is cost-minimizing to concentrate all production on the plant/s with the smallest $\beta_j < 0$, since it benefits from the most rapidly decreasing average costs. Figure 4 depicts a firm with plant 1 (2) having increasing (decreasing, respectively) average costs.

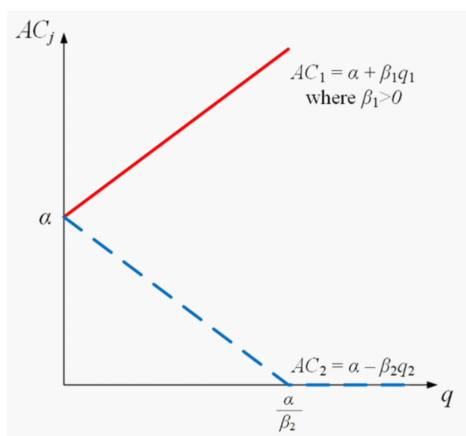

Figure 4. $\beta_1 > 0$ and $\beta_2 < 0$.