Recitation #4 (09/25/2020)- Solution

1. An individual consumes only good 1 and 2, and his preferences over these two goods can be represented by Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^\alpha x_2^\beta \quad \text{where } \alpha, \beta > 0.$$

(For generality, we do not impose any assumptions on the sum of the exponents, i.e., $\alpha + \beta$ can satisfy $\alpha + \beta > 1$ or $\alpha + \beta < 1$.) This individual currently works for a firm in a city where initial prices are $p^0 = (p_1, p_2)$, and his wealth is w.

(a) Find the Walrasian demand for goods 1 and 2 of this individual, $x_1(p, w)$ and $x_2(p, w)$.

- We know that the Walrasian demand in the Cobb-Douglas case are $x_1(p, w) = \frac{\alpha}{\alpha+\beta}\frac{w}{p_1}$ and $x_2(p, w) = \frac{\beta}{\alpha+\beta}\frac{w}{p_2}$; as shown in previous chapters. As a practice we next demonstrate this result again.
- The Lagrangian of this UMP is

$$\mathcal{L}(x_1, x_2; \lambda) = x_1^\alpha x_2^\beta - \lambda \left[p_1 x_1 + p_2 x_2 - w\right]$$

The first order conditions are:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 = 0\end{aligned}$$

Solving for λ on both first order conditions, we obtain

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{p_1} = \frac{\beta x_1^\alpha x_2^{\beta-1}}{p_2} \Longleftrightarrow x_2 = \frac{\beta p_1 x_1}{\alpha p_2}$$

Using the budget constraint (which is binding), we have

$$p_1 x_1 + p_2 x_2 = w \Longleftrightarrow x_1 = \frac{w}{p_1} - \frac{p_2 x_2}{p_1}$$

and plugging this expression of x_2 we found above, yields the Walrasian demand for good 1

$$x_1 = \frac{w}{p_1} - \frac{p_2 \left(\frac{\beta p_1 x_1}{\alpha p_2}\right)}{p_1} \Longleftrightarrow x_1 = \frac{\alpha w}{(\alpha + \beta)\, p_1}$$

and, hence, the Walrasian demand for good 2 is

$$x_2 = \frac{\beta p_1 \left(\frac{\alpha w}{(\alpha+\beta)p_1}\right)}{\alpha p_2} = \frac{\beta w}{(\alpha+\beta)\, p_2}$$

Hence, the Walrasian demand function is

$$x_1(p,w) = \frac{\alpha w}{(\alpha+\beta)\, p_1} \quad \text{and} \quad x_2(p,w) = \frac{\beta w}{(\alpha+\beta)\, p_2}$$

(b) Find his indirect utility function at price vector p, and denote it as $v(p,w)$.

- Plugging the above Walrasian demand functions in the consumer's utility function, we obtain

$$\begin{aligned} v(p,w) &= \left[\frac{\alpha w}{(\alpha+\beta)\, p_1}\right]^{\alpha} \left[\frac{\beta w}{(\alpha+\beta)\, p_2}\right]^{\beta} \\ &= \left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta} \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{\beta}{p_2}\right)^{\beta} \end{aligned}$$

(c) The firm that this individual works for is considering moving its office to a different city, where good 1 has the same price, but good 2 (e.g., housing) is twice as expensive, i.e., the new price vector is $p' = (p_1, 2p_2)$. Find the value of the indirect utility function in the new location. Let us denote this indirect utility function $v(p', w)$.

- The indirect utility function $v(p', w)$ is

$$v(p', w) = \left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta} \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{\beta}{2p_2}\right)^{\beta}$$

where, relative to $v(p,w)$, only the price of good 2 has changed (namely, it has doubled), while all other elements remain unaffected.

(d) This individual's expenditure function is[1]

$$e(p,u) = (\alpha+\beta) \left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

Evaluate this expenditure function in the following cases:

[1] As a practice, you can set up the consumer's expenditure minimization problem (EMP), find the Hicksian demands that emerge from solving this EMP, $h_1(p,u)$ and $h_2(p,u)$, and afterwards plug them into $p_1x_1 + p_2x_2$ to obtain the expenditure function $e(p,u) \equiv p_1h_1(p,u) + p_2h_2(p,u)$. After some algebra, you should find an expression of $e(p,u)$ that coincides with that provided in the exercise.

1. Under initial prices, p, and maximal utility level $u \equiv v(p, w)$, and denote it by $e(p, u)$.

$$e(p,u) = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\underbrace{\left[\left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}}}_{u} = w$$

2. Under initial prices, p, and maximal utility level $u' \equiv v(p', w)$, and denote it by $e(p, u')$.

$$e(p,u') = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left[\left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{2p_2}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} = \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}w$$

3. Under new prices, p', and maximal utility level $u \equiv v(p, w)$, and denote it by $e(p', u)$.

$$e(p',u) = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{2p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left[\left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} = 2^{\frac{\beta}{\alpha+\beta}}w$$

4. Under new prices, p', and maximal utility level $u' \equiv v(p', w)$, and denote it by $e(p', u')$.

$$e(p',u') = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{2p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left[\left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{2p_2}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} = w$$

(e) Find this individual's equivalent variation due to the price change. Explain how your result can be related with this proposal of the worker to his boss: "I would really prefer to stay in this city. In fact, I would accept a salary reduction if I could keep working for the firm in this city."

- The equivalent variation of a price change is given by

$$EV = e(p', u') - e(p, u')$$

using the results from the previous part, we have that $e(p', u') = w$, while $e(p, u') = \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}w$, thus implying that the equivalent variation is

$$EV = w - \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}w$$

That is, this individual would be willing to accept a reduction in his wealth of $w - \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}w$ in order to avoid moving to a different city. [Alternatively, the individual is willing to accept a reduction of $\left(1 - \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}\right)$% of his wealth.] Figure 1 depicts the equivalent variation for the case in which $\alpha = \beta = \frac{1}{2}$, i.e., $EV = w\left(1 - \frac{1}{\sqrt{2}}\right)$. In particular, the figure depicts the wealth level of this individual, w, in the 45^0-line; and the equivalent variation (in the shaded area). Hence, the unshaded region below the 45^0-line represents the remaining income that this individual would retain after giving up the amount found in the equivalent variation.

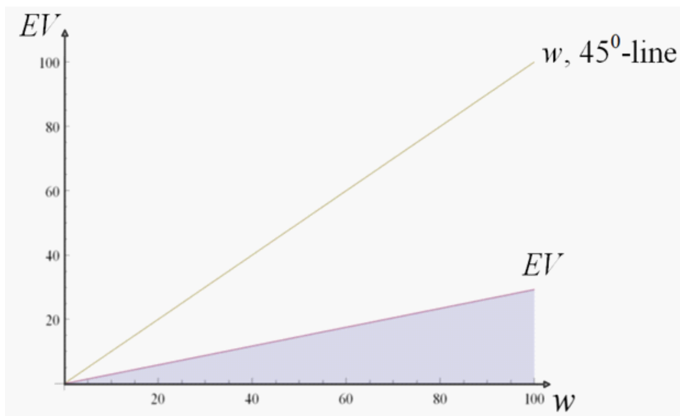

Figure 1. Equivalent variation (shaded area)..

(f) How is this individual's consumer surplus affected by the price change? (The change in consumer surplus is often referred to as the "area variation (AV)"

- The area variation is given by the area below the Walrasian demand of good 2 (since only the price of this good changes), between the initial and final price level. That is,

$$AV = \int_{p_2}^{2p_2} x_2(p, w)dp = \int_{p_2}^{2p_2} \frac{\beta}{(\alpha+\beta)\,p} w \; dp$$

and rearranging

$$= \frac{\beta}{(\alpha+\beta)} w \int_{p_2}^{2p_2} \frac{1}{p} \; dp = \frac{\beta}{(\alpha+\beta)} w \ln 2$$

Hence, moving to the new city would imply a reduction in this individual's welfare of $\frac{\beta}{(\alpha+\beta)} w \ln 2$, or $\left(\frac{\beta}{(\alpha+\beta)} \ln 2\right)$% of his wealth. Figure 2 depicts the AV for the case in which $\alpha = \beta = \frac{1}{2}$, i.e., $AV = \frac{\ln 2}{2} w$, and compares it with the EV found in part (e) of the exercise.

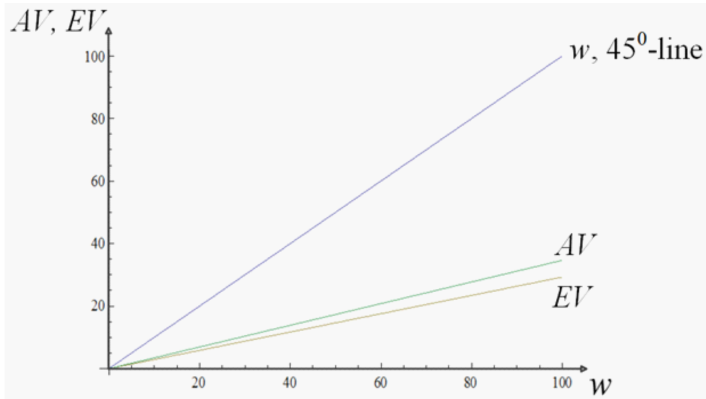

Figure 2. Area variation and equivalent variation.

(g) Which of the previous welfare measures in questions (e) and (f) coincide? Which of them do *not* coincide? Explain.

- None of them coincide, since this individual's preferences produces a positive income effect.

(h) Consider how the welfare measures from questions (e) and (f) would be modified if this individual's preferences were represented, instead, by the utility function $v(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$.

- Since we have just applied a monotonic transformation to the initial utility function, $u(x_1, x_2)$, the new utility function $v(x_1, x_2)$ represents the same preference relation as utility function $v(x_1, x_2)$. Hence, the welfare results that we would obtain from function $v(x_1, x_2)$ would be the same as those with utility function $u(x_1, x_2)$. This is, in fact, one of the advantages of using monetary measures of welfare change (such as the equivalent, compensating, or area variation) rather than the simple difference in utility levels before and after the price change, i.e., $u' - u$. In particular, while the monetary measures are insensitive to monotonic transformations of the utility function, the utility difference when the consumer has utility function $u(x)$, i.e., $u' - u$, may differ from that when his utility experiences a monotonic transformation, $v' - v$.