# Homework 1 - EconS 501 (Wednesday, September 2nd)

1. Moana and Maui need to find the magical fish hook. Maui lost this weapon after stealing the heart of Te Fiti and his subsequent battle with the lava demon Te Kā. The fish hook was lost in sea, and eventually was found by Maui's arch-rival Tamatoa, who placed the fish hook on his shell as a prize. In order to find the hook they need to combine two techniques of navigation, that is, intense observation of (1) the celestial bodies in the sky (technique $x$) and (2) the swells of the water (technique $y$). Maui is an expert in the art of navigation, and he weakly prefers a combination of technique $x$ and $y$ that contains more of observation of the sky, i.e., $(x_1, y_1) \succsim (x_2, y_2)$ if and only if $x_1 \geqslant x_2 + 1$. For this preference relation: describe the upper contour set, the lower contour set, the indifference set of bundle $(3, 2)$, and interpret them. Then check whether this preference relation is rational (by separately examining whether they are complete and transitive), monotone, and convex.

   (a) Bundle $(x_1, y_1)$ is weakly preferred to $(x_2, y_2)$, i.e., $(x_1, y_1) \succsim (x_2, y_2)$ if and only if $x_1 \geqslant x_2 + 1$.

   - Let us first build some intuition on this preference relation. First, note that an individual prefers a bundle $x$ to another bundle $y$ if and only if the first component of bundle $x$, $x_1$, contains at least one unit more than the first component of bundle $y$, i.e., $x_1 \geqslant y_1 + 1$. For instance, (3, 2) is preferred to (1, 2) since $x_1 = 3$ and $x_2 = 1$, thus implying $3 \geqslant 1 + 1 = 2$. Importantly, the individual ignores the content of the second component when comparing two bundles. Let us next describe the upper contour, lower contour, and indifference set of a given bundle, such as (3,2). The upper contour set of this bundle is given by

     $$UCS(3, 2) = \{(x_1, y_1) \succsim (3, 2) \iff x_1 \geq 3 + 1\} = \{(x_1, x_2) : x_1 \geq 4\}$$

     while the lower contour set is defined as

     $$LCS(3, 2) = (3, 2) \succsim (x_1, y_1) \iff x_1 \leq 2$$

     Finally, the consumer is indifferent between bundle (3,2) and the set of bundles where

     $$IND(3, 2) = \{(x_1, y_1) \sim (3, 2) \iff 2 < x_1 < 4\}$$

   - *Completeness.* For this property to hold, we need that, for any pair of bundles

$(x_1, y_1)$ and $(x_2, y_2)$, either $(x_1, y_1) \succsim (x_2, y_2)$ or $(x_2, y_2) \succsim (x_1, y_1)$, or both (i.e., $(x_1, y_1) \sim (x_2, y_2)$). Since this preference relation only depends on the first component of every bundle, we have that, for every pair of bundles $(x_1, y_1)$ and $(x_2, y_2)$, either:

1. $x_1 \geq x_2 + 1$, which implies that $(x_1, y_1) \succsim (x_2, y_2)$; or
2. $x_1 < x_2 + 1$, which implies

$$x_2 > x_1 - 1 > x_1 + 1,$$

   and hence $x_2 > x_1 + 1$, thus ultimately yielding $(x_2, y_2) \succsim (x_1, y_1)$. Hence, this preference relation is complete.

Additionally, note that this preference relation satisfies reflexivity, since completeness implies reflexivity, i.e., every bundle $(x_1, y_1)$ is weakly preferred to itself.

- *Transitivity.* We need to show that, for any three bundles $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ such that

$$(x_1, y_1) \succsim (x_2, y_2) \text{ and } (x_2, y_2) \succsim (x_3, y_3), \text{ then } (x_1, y_1) \succsim (x_3, y_3)$$

This property holds for this preference relation. In order to show this result, notice that a bundle $(x_1, y_1)$ is preferred to another bundle $(x_2, y_2)$ if its first component, $x_1$, is larger than that of the other bundle, $x_2$, by more than one unit, i.e., condition $x_1 \geq x_2 + 1$ is equivalent to $1 \leq x_1 - x_2$. A similar argument can be extended to the comparison between two bundles $(x_2, y_2)$ and $(x_3, y_3)$, where the former is preferred to the latter if and only if the distance between their first components is greater than one, i.e., $1 \leq x_2 - x_3$. Hence, for bundle $(x_1, y_1)$ to be preferred to $(x_3, y_3)$, i.e., $(x_1, y_1) \succsim (x_3, y_3)$, we need that the distance between their first components is greater than one, i.e., $1 \leq x_1 - x_3$; as we next show with an example.
Consider the following three bundles (notice that the second component of every bundle is inconsequential, since the preference ordering only relies on a comparison of the first component of every vector):

$$\begin{aligned} (x_1, y_1) &= (6, 4) \\ (x_2, y_2) &= (5, 1) \\ (x_3, y_3) &= (4, 2) \end{aligned}$$

First, note that $(x_1, y_1) \succsim (x_2, y_2)$ since the difference in their first component is greater (or equal) to one unit, $x_1 \geq x_2 + 1$ (i.e., $6 \geq 5 + 1$). Additionally, $(x_2, y_2) \succsim (x_3, y_3)$ is also satisfied since $x_2 \geq x_3 + 1$ (i.e., $5 \geq 4 + 1$). Therefore, $(x_1, y_1) \succsim (x_3, y_3)$ since the difference between $x_1$ and $x_3$ is larger than one unit, $x_1 \geq x_3 + 1$. Hence, this preference relation satisfies transitivity.

- *Monotonicity.* This property is satisfied for this preference relation. In particular, increasing the amount of good 1 yields a new bundle $(x_1 + \varepsilon, y_1)$ that is weakly preferred to the original bundle $(x_1, y_1)$, i.e., the comparison of their first component yields $x_1 + \varepsilon \geq x_1 + 1$, which holds iff $\varepsilon \geq 1$. Similarly, increasing the amount of the second component produces a new bundle $(x_1, y_1 + \varepsilon)$ which is weakly preferred to the original bundle $(x_1, y_1)$. Recall that this individual compares bundles by evaluating the first component alone. Since in this case the amount of the first component is unaffected, then he is indifferent between bundle $(x_1, y_1)$ and $(x_1, y_1 + \varepsilon)$; an indifference that is allowed by the definition of monotonicity. Hence, the preference relation does not satisfies monotonicity.
- *Convexity.* This property implies that the upper contour set must be convex, that is, if bundle $(x_1, y_1)$ is weakly preferred to $(x_2, y_2)$, $(x_1, y_1) \succsim (x_2, y_2)$, then the convex combination of these two bundles is also weakly preferred to $(x_2, y_2)$,

$$\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \succsim (x_2, y_2) \quad \text{for any } \lambda \in [0, 1]$$

In this case, $(x_1, y_1) \succsim (x_2, y_2)$ implies that $x_1 \geq x_2 + 1$; whereas $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \succsim (x_2, y_2)$ implies

$$\lambda x_1 + (1 - \lambda) x_2 \geq x_2 + 1$$

which simplifies to $\lambda x_1 \geq \lambda x_2 + 1$. However, the premise from $(x_1, y_1) \succsim (x_2, y_2)$, i.e., $x_1 \geq x_2 + 1$, entails that $\lambda x_1 \geq \lambda x_2 + 1$ must also hold. (To see that, note that $x_1 \geq x_2 + 1$ can be written as $(x_1 - x_2) - 1 \geq 0$, while $\lambda x_1 \geq \lambda x_2 + 1$ can be expressed as $\lambda (x_1 - x_2) - 1 \geq 0$, where $\lambda (x_1 - x_2) - 1 \leq (x_1 - x_2) - 1$ since $\lambda \in [0, 1]$.) Therefore, $(x_1 - x_2) - 1 \geq 0$ is not a sufficient condition for $\lambda (x_1 - x_2) - 1 \geq 0$. We also need a more restrictive condition $(x_1 - x_2) \geq 2$. Hence, this preference relation is not convex, since $(x_1 - x_2) \geq 1$. Example, consider (3,2) and (2,2), in this case $(x_1, y_1) \succsim (x_2, y_2)$, however, $0.5 \times 3 + (1 - 0.5)\, 2 \ngeq 2 + 1$.

2. Consider the following preference relation defined in $X = \mathbb{R}^2_+$. A bundle $(x_1, x_2)$ is weakly preferred to another bundle $(y_1, y_2)$, i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3y_1 + 2y_2, 2y_1 + 3y_2\}$$

(a) For any given bundle $(y_1, y_2)$, draw the upper contour set, the lower contour set, and the indifference set of this preference relation. Interpret.

- Take a bundle $(2, 1)$. Then,

$$\min\{3 * 2 + 2 * 1, 2 * 2 + 3 * 1\} = \min\{8, 7\} = 7.$$

The upper contour set of this bundle is given by

$$\begin{aligned} UCS(2,1) &= \{(x_1, x_2) \succsim (2,1)\} \\ &= \{\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq 7 \equiv \min\{8, 7\}\} \end{aligned}$$

which is graphically represented by all those bundles in $\mathbb{R}^2_+$ which are strictly above *both* lines $3x_1 + 2x_2 = 7$ and $2x_1 + 3x_2 = 7$. That is, for all $(x_1, x_2)$ strictly above both lines

$$x_2 = \frac{7}{2} - \frac{3}{2}x_1 \text{ and } x_2 = \frac{7}{3} - \frac{2}{3}x_1.$$

(See figure 1, which depicts these two lines and shades the set of bundles lying weakly above both lines.)

- On the other hand, the lower contour set is defined as

$$\begin{aligned} LCS(2,1) &= \{(2,1) \succsim (x_1, x_2)\} \\ &= \{7 \geq \min\{3x_1 + 2x_2, 2x_1 + 3x_2\}\}, \end{aligned}$$

which is graphically represented by all bundles $(x_1, x_2)$ weakly below the maximum of the lines described above. For instance, bundle $(y_1, y_2) = (2.5, 0)$, which lies on the horizontal axis and between both lines' horizontal intercept, implies

$$\min\{3 \cdot 2.5 + 2 \cdot 0, 2 \cdot 2.5 + 3 \cdot 0\} = \min\{7.5, 5\} = 5$$

thus implying that this consumer prefers bundle $(x_1, x_2) = (2, 1)$ than $(y_1, y_2) = (2.5, 0)$. A similar argument applies to all other bundles lying above $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ and below $x_2 = \frac{7}{3} - \frac{2}{3}x_1$, where bundle $(2.5, 0)$ also belongs; see the triangle that both lines form at the right-hand side of the figure. Similarly, bundles such as $(0, 2.5)$ yield

$$\min\{3 \cdot 0 + 2 \cdot 2.5, 2 \cdot 0 + 3 \cdot 2.5\} = \min\{5, 7.5\} = 5,$$

which implies that the consumer also prefers bundle $(2, 1)$ to $(0, 2.5)$. An analogous argument applies to all bundles above line $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ but below $x_2 = \frac{7}{3} - \frac{2}{3}x_1$ in the triangle at the left-hand side of figure 1.

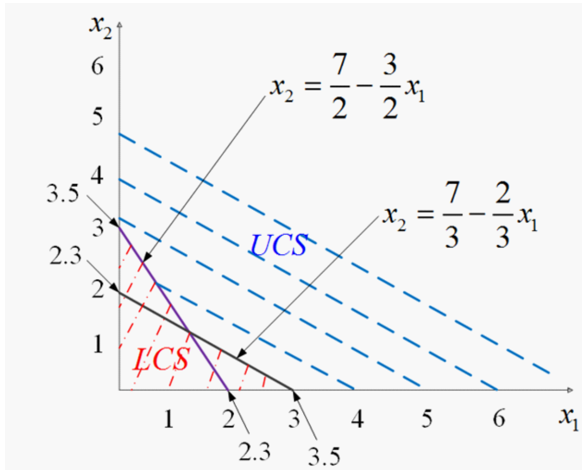


Figure 1. UCS and LCS of bundle (2,1).

Finally, those bundles for which the UCS and LCS overlap are those in IND of bundle (2,1).

(b) Check if this preference relation satisfies: (i) completeness, (ii) transitivity, and (iii) weak convexity.

- *Completeness.* First, note that both of the elements in the $\min\{\cdot\}$ operator are real numbers, i.e., $(3x_1 + 2x_2) \in \mathbb{R}_+$ and $(2x_1 + 3x_2) \in \mathbb{R}_+$, thus implying that the minimum

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} = a$$

exists and it is also a real number, $a \in \mathbb{R}_+$. Similarly, the minimum

$$\min\{3y_1 + 2y_2, 2y_1 + 3y_2\} = b$$

exists and is a real number, $b \in \mathbb{R}_+$. Therefore, we can easily compare $a$ and $b$, obtaining that either $a \geq b$, which implies $(x_1, x_2) \succsim (y_1, y_2)$; or $a \leq b$, which implies $(y_1, y_2) \succsim (x_1, x_2)$, or both, $a = b$, which entails $(x_1, x_2) \sim (y_1, y_2)$.

Hence, the preference relation is complete.

- *Transitivity.* We need to show that, for any three bundles $(x_1, x_2)$, $(y_1, y_2)$ and $(z_1, z_2)$ such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ implies

$$a \equiv \min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \equiv b$$

and $(y_1, y_2) \succsim (z_1, z_2)$ implies that

$$b \equiv \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \geq \min\{3z_1 + 2z_2, 2z_1 + 3z_2\} \equiv c$$

Combining both conditions we have that $a \geq b \geq c$, which implies that $a \geq c$. Hence, we have that

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3z_1 + 2z_2, 2z_1 + 3z_2\}$$

and thus $(x_1, x_2) \succsim (z_1, z_2)$, implying that this preference relation is transitive.

- *Weak Convexity.* This property implies that the upper contour set must be convex. That is, if bundle $(x_1, x_2)$ is weakly preferred to $(y_1, y_2)$, $(x_1, x_2) \succsim (y_1, y_2)$, then the convex combination of these two bundles is also weakly preferred to $(y_1, y_2)$,

$$\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2) \succsim (y_1, y_2) \quad \text{for any } \lambda \in [0, 1]$$

For compactness, let $a \equiv 3x_1 + 2x_2$, $b \equiv 2x_1 + 3x_2$, $c \equiv 3y_1 + 2y_2$ and $d \equiv 2y_1 + 3y_2$. Hence, the property that $(x_1, x_2) \succsim (y_1, y_2)$ implies $\min\{a, b\} \geq \min\{c, d\}$. We therefore need to show that

$$\min\{\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d\} \geq \min\{c, d\}$$

1. *First case*: $\min\{a, b\} = a$, $\min\{c, d\} = c$ and $a \geq c$. Therefore,

$$\min\{\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d\} = \lambda a + (1-\lambda)c$$

   and $\lambda a + (1-\lambda)c > \min\{c, d\} = c$. For this case, convexity is satisfied.

2. *Second case*: $\min\{a, b\} = a$, $\min\{c, d\} = d$ and $a \geq d$. Hence, $a > b$ and

$c > d$, implying that

$$\min\{\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d\} = \lambda a + (1-\lambda)d$$

and $\lambda a + (1-\lambda)d \geq \min\{c,d\} = d$ given that $a \geq d$. For this case, convexity is satisfied as well. An analogous argument applies in the other two cases, in which $\min\{a,b\} = b$ and $\min\{c,d\} = c$, and in which $\min\{a,b\} = b$ but $\min\{c,d\} = d$.

3. Explain transitivity in preference relations. Provide an example (different to the examples discussed in class) where this property is not satisfied and discuss the consequences of intransitive preferences.

4. For each of the following preference relations in the consumption of two goods (1 and 2): describe the upper contour set, the lower contour set, the indifference set of bundle (2,1), and interpret them. Then check whether these preference relations are rational (by separately examining whether they are complete and transitive), monotone, and convex.

   (a) Bundle $(x_1, x_2)$ is weakly preferred to $(y_1, y_2)$, i.e., $(x_1, x_2) \succsim (y_1, y_2)$ if and only if $x_1 \geqslant y_1 - 1$.

   - Let us first build some intuition on this preference relation. First, note that an individual prefers a bundle $x$ to another bundle $y$ if and only if the first component of bundle $x$, $x_1$, contains at least one unit less than the first component of bundle $y$, i.e., $x_1 \geqslant y_1 - 1$. For instance, (2, 1) is preferred to (2, 6) since $x_1 = 2$ and $y_1 = 2$, thus implying $2 \geqslant 2 - 1 = 1$. Importantly, the individual ignores the content of the second component when comparing two bundles. Let us next describe the upper contour, lower contour, and indifference set of a given bundle, such as (2, 1). You can take any other bundle of course! The upper contour set of this bundle is given by

   $$UCS(2,1) = \{(x_1, x_2) \succsim (2,1) \iff x_1 \geq 2 - 1\} = \{(x_1, x_2) : x_1 \geq 1\}$$

   while the lower contour set is defined as

   $$LCS(2,1) = \{(2,1) \succsim (x_1, x_2) \iff 2 \geq x_1 - 1\} = \{(x_1, x_2) : x_1 \leq 3\}$$

   Finally, the consumer is indifferent between bundle (2,1) and the set of bundles where

$$IND(2,1) = \{(x_1, x_2) \sim (2,1) \iff 1 \leq x_1 \leq 3\}$$

Figure 2 depicts:

1. All bundles in $\mathbb{R}^2_+$ such that $x_1 \geq 1$, and thus belong to the $UCS(2,1)$, i.e., the set of bundles that are weakly preferred to (2,1);
2. All bundles such that $x_1 \leq 3$ and are therefore in the $LCS(2,1)$, i.e., the set of bundles weakly preferred by (2,1); and
3. Those bundles in between, $1 \leq x_1 \leq 3$, in the $IND(2,1)$ are indifferent between them and bundle (2,1). [Figure 2 represents the UCS, LCS and IND sets, where all cutoffs we found were on the $x_1-$axis, since this individual ignores the amount of $x_2$.]

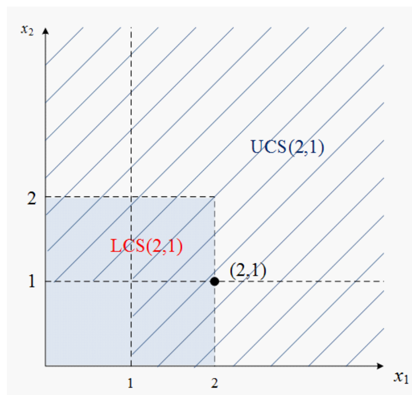


Figure 2. UCS, LCS, and IND of bundle (2,1).

As a remark, this preference relation satisfies *continuity*. In particular, continuity requires that both the upper and the lower contour sets are closed, which is satisfied given that they both contain their boundary points.

- *Completeness.* For this property to hold, we need that, for any pair of bundles $(x_1, x_2)$ and $(y_1, y_2)$, either $(x_1, x_2) \succsim (y_1, y_2)$ or $(y_1, y_2) \succsim (x_1, x_2)$, or both (i.e., $(x_1, x_2) \sim (y_1, y_2)$). Since this preference relation only depends on the first component of every bundle, we have that, for every pair of bundles $(x_1, x_2)$ and $(y_1, y_2)$, either:

  1. $x_1 \geq y_1 - 1$, which implies that $(x_1, x_2) \succsim (y_1, y_2)$; or

2. $x_1 < y_1 - 1$, which implies

$$y_1 > x_1 + 1 > x_1 - 1,$$

and hence $y_1 > x_1 - 1$, thus ultimately yielding $(y_1, y_2) \succsim (x_1, x_2)$. Hence, this preference relation is complete.

Additionally, note that this preference relation satisfies reflexivity, since completeness implies reflexivity, i.e., every bundle $(x_1, x_2)$ is weakly preferred to itself.

- *Transitivity.* We need to show that, for any three bundles $(x_1, x_2)$, $(y_1, y_2)$ and $(z_1, z_2)$ such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

This property does not hold for this preference relation. In order to show this result, notice that a bundle $(x_1, x_2)$ is preferred to another bundle $(y_1, y_2)$ if its first component, $x_1$, is larger than that of the other bundle, $y_1$, by less than one unit, i.e., condition $x_1 \geq y_1 - 1$ is equivalent to $1 \geq y_1 - x_1$. This condition about the distance between $x_1$ and $y_1$ is depicted in the bottom left-hand side of figure 2. A similar argument can be extended to the comparison between two bundles $(y_1, y_2)$ and $(z_1, z_2)$, where the former is preferred to the latter if and only if the distance between their first components is smaller than one, i.e., $1 \geq z_1 - y_1$; also depicted at the bottom of figure 2, but on the right-hand side. Hence, for bundle $(x_1, x_2)$ to be preferred to $(z_1, z_2)$, i.e., $(x_1, x_2) \succsim (z_1, z_2)$, we need that the distance between their first components is smaller than one, i.e., $1 \geq z_1 - x_1$; as we next show with a counterexample. Consider the following three bundles (notice that the second component of every bundle is inconsequential, since the preference ordering only relies on a comparison of the first component of every vector):

$$\begin{aligned} (x_1, x_2) &= (5, 4) \\ (y_1, y_2) &= (6, 1) \\ (z_1, z_2) &= (7, 2) \end{aligned}$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ since the difference in their first component is smaller (or equal) to one unit, $x_1 \geq y_1 - 1$ (i.e., $5 \geq 6 - 1$). Additionally, $(y_1, y_2) \succsim (z_1, z_2)$ is also satisfied since $y_1 \geq z_1 - 1$ (i.e., $6 \geq 7 - 1$). However, $(x_1, x_2) \not\succsim (z_1, z_2)$ since the difference between $z_1$ and $x_1$ is larger than one unit, $x_1 \not\geq z_1 - 1$ (i.e., $5 \not\geq 7 - 1$). Hence, this preference relation does not satisfy transitivity.

- *Monotonicity.* This property is satisfied for this preference relation. In particular, increasing the amount of good 1 yields a new bundle $(x_1 + \varepsilon, x_2)$ that is weakly preferred to the original bundle $(x_1, x_2)$, i.e., the comparison of their first component yields $x_1 + \varepsilon \geq x_1 - 1$, or $\varepsilon \geq -1$, which holds since $\varepsilon > 0$ by assumption. Similarly, increasing the amount of the second component produces a new bundle $(x_1, x_2 + \varepsilon)$ which is weakly preferred to the original bundle $(x_1, x_2)$. Recall that this individual compares bundles by evaluating the first component alone. Since in this case the amount of the first component is unaffected, then he is indifferent between bundle $(x_1, x_2)$ and $(x_1, x_2 + \varepsilon)$; an indifference that is allowed by the definition of monotonicity. Hence, the preference relation satisfies monotonicity. As a curiosity, note that, while the preference relation satisfies monotonicity, it does not satisfy strong monotonicity. Indeed, for this property to hold, we need that an increase in the amount of *any* of the goods yields a new bundle that is strictly preferred to the original bundle $(x_1, x_2)$. While this is true if we increase the amount of good 1, it is not if we only increase the amount of good 2, thus not satisfying strict monotonicity.
- *Convexity.* This property implies that the upper contour set must be convex, that is, if bundle $(x_1, x_2)$ is weakly preferred to $(y_1, y_2)$, $(x_1, x_2) \succsim (y_1, y_2)$, then the convex combination of these two bundles is also weakly preferred to $(y_1, y_2)$,

  $$\lambda(x_1, x_2) + (1 - \lambda)(y_1, y_2) \succsim (y_1, y_2) \quad \text{for any } \lambda \in [0, 1]$$

  In this case, $(x_1, x_2) \succsim (y_1, y_2)$ implies that $x_1 \geq y_1 - 1$; whereas $\lambda(x_1, x_2) + (1 - \lambda)(y_1, y_2) \succsim (y_1, y_2)$ implies

  $$\lambda x_1 + (1 - \lambda) y_1 \geq y_1 - 1$$

  which simplifies to $\lambda x_1 \geq \lambda y_1 - 1$. However, the premise from $(x_1, x_2) \succsim (y_1, y_2)$, i.e., $x_1 \geq y_1 - 1$, entails that $\lambda x_1 \geq \lambda y_1 - 1$ must also hold. (To see that, note that $x_1 \geq y_1 - 1$ can be written as $(x_1 - y_1) + 1 \geq 0$, while $\lambda x_1 \geq \lambda y_1 - 1$ can be expressed as $\lambda (x_1 - y_1) + 1 \geq 0$, where $\lambda (x_1 - y_1) + 1 \leq (x_1 - y_1) + 1$ since $\lambda \in [0, 1]$.) Therefore, $(x_1 - y_1) + 1 \geq 0$ is a sufficient condition for $\lambda (x_1 - y_1) + 1 \geq 0$, ultimately implying that $\lambda x_1 \geq \lambda y_1 - 1$ must hold. Hence, this preference relation is convex.

(b) Bundle $(x_1, x_2)$ is weakly preferred to $(y_1, y_2)$, i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if $x_1 \geq y_1 - 1$ and $x_2 \leq y_2 + 1$.

- Let us first build some intuition on this preference relation. Similarly as the preference relation we first analyzed, the individual prefers bundle $x$ to $y$ if the first component of $x$ is larger than that of $y$ in less than one unit, i.e., $x_1 \geqslant y_1 + 1$ or $1 \geqslant y_1 - x_1$; but in addition, he must find that the difference between their second components is larger than one unit, i.e., $x_2 \leqslant y_2 + 1$ or $1 \leqslant y_2 - x_2$.
- Let us next find the upper contour, lower contour, and indifference set of a given bundle, such as (2, 1). The upper contour set of this bundle is given by

$$\begin{aligned} UCS(2,1) &= \{(x_1, x_2) \succsim (2,1) \iff x_1 \geq 2 - 1 \text{ and } x_2 \leq 1 + 1\} \\ &= \{(x_1, x_2) : x_1 \geq 1 \text{ and } x_2 \leq 2\} \end{aligned}$$

which is graphically represented in figure 3 by all those bundles in the lower right-hand corner (below $x_2 = 2$ and to the right of $x_1 = 1$). On the other hand, the lower contour set is defined as

$$\begin{aligned} LCS(2,1) &= \{(2,1) \succsim (x_1, x_2) \iff 2 \geq x_1 - 1 \text{ and } 1 \leq x_2 + 1\} \\ &= \{(x_1, x_2) : x_1 \leq 3 \text{ and } x_2 \geq 0\} \end{aligned}$$

Finally, the consumer is indifferent between bundle (2,1) and the set of bundles where

$$IND(2,1) = \{(x_1, x_2) \sim (2,1) \iff 1 \leq x_1 \leq 3 \text{ and } 0 \leq x_2 \leq 2\}$$

- *Completeness.* From the above analysis it is easy to note that completeness is *not* satisfied, since there are bundles in the area $x_1 > 3$ and $x_2 > 2$ where our preference relation does not specify if they belong to the upper contour set, the lower contour set, or the indifference set of bundle (2, 1). Hence, any bundle in the unshaded region where $x_1 > 3$ and $x_2 > 2$ would be incomparable with (2, 1). Another way to prove that completeness does not hold is by finding a counterexample. In particular, we must find an example of two bundles such that neither $(x_1, x_2) \succsim (y_1, y_2)$ nor $(y_1, y_2) \succsim (x_1, x_2)$. Let us take, for instance, two bundles,

$$(x_1, x_2) = (1, 2) \text{ and } (y_1, y_2) = (4, 6)$$

We have that:

1. $(x_1, x_2) \not\succsim (y_1, y_2)$ since $1 \ngeq 4 - 1$ for the first component of the bundle (and we need $x_1 \geq y_n - 1$ for $(x_1, x_2) \succsim (y_1, y_2)$ to hold), and
2. $(y_1, y_2) \not\succsim (x_1, x_2)$ since $6 \nleq 2+1$ for the second component of the bundle. Hence, for these are two bundles neither $(x_1, x_2) \succsim (y_1, y_2)$ nor $(y_1, y_2) \succsim (x_1, x_2)$, which implies that this preference relation is not complete.

- *Transitivity.* We need to show that, for any three bundles $(x_1, x_2)$, $(y_1, y_2)$ and $(z_1, z_2)$ such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

This property does not hold for this preference relation. In order to show that, let us consider the following three bundles (that is, we are finding a counterexample to show that transitivity does not hold):

$$\begin{aligned} (x_1, x_2) &= (2, 1) \\ (y_1, y_2) &= (3, 4) \\ (z_1, z_2) &= (4, 6) \end{aligned}$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ since the distance between their first components is not larger than one unit $x_1 \geq y_1 - 1$ (i.e., $2 \geq 3 - 1$), and the distance between the second components is larger than one unit $x_2 \leq y_2 + 1$ (i.e., $1 \leq 4 + 1$). Additionally, $(y_1, y_2) \succsim (z_1, z_2)$ is also satisfied since $y_1 \geq z_1 - 1$ for the first component (i.e., $3 \geq 4 - 1$), and $y_2 \leq z_2 + 1$ for the second component (i.e., $3 \leq 4+1$). However, $(x_1, x_2) \not\succsim (z_1, z_2)$ since the difference of the first components is strictly larger than one unit $x_1 \ngeq z_1 - 1$ (i.e., $2 \ngeq 4 - 1$). Hence, this preference relation does not satisfy Transitivity.

- *Monotonicity.* For this property to hold, we need that an increase in the amounts of one good yields a new bundle that is weakly preferred to the original bundle. Indeed, if we increase the amount of good 1 by $\varepsilon > 0$ to create bundle $(x_1 + \varepsilon, x_2)$, we have that the first component satisfies $x_1 + \varepsilon \geqslant x_1 - 1$, i.e., $\varepsilon \geqslant -1$, and the second component satisfies $x_2 \leqslant x_2 + 1$, i.e., $0 \leqslant 1$. If we only increase the amounts of good 2, a similar argument applies. Finally, if we increase the amounts of both goods 1 and 2 simultaneously, according to the definition of monotonicity we need that the newly created bundle is strictly preferred to the initial bundle, i.e., $(x_1 + \varepsilon, x_2 + \delta) \succ (x_1, x_2)$ where constants $\varepsilon, \delta > 0$ are allowed to differ for each good. For this relationship to hold, note that we need that: (1) the first components satisfy $x_1+\varepsilon \geq x_1-1$, or $\varepsilon \geq -1$ (which holds by definition); and (2) the second components satisfy $x_2+\delta \leq x_2+1$, which implies $\delta \leq 1$ (which does not necessarily hold by assumption). Therefore, for this preference relation to be monotonic, we need that $\delta \leq 1$. In other words, if good 2 is increased by more than one unit, the preference relation is not monotonic.

For instance, if the amount of both goods is increased by two units, i.e., $\varepsilon = \delta = 2$, the new bundle $(x_1+2, x_2+2)$ is not necessarily preferred to the original bundle $(x_1, x_2)$ since the condition on the first component, $x_1+2 \geq x_1-1$, holds but that on the second component, $x_2+2 \leq x_2+1$, does not.

- *Convexity.* This property implies that the upper contour set must be convex. That is, if bundle $(x_1, x_2)$ is weakly preferred to $(y_1, y_2)$, $(x_1, x_2) \succsim (y_1, y_2)$, then the convex combination of these two bundles is also weakly preferred to $(y_1, y_2)$,

$$\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2) \succsim (y_1, y_2) \quad \text{for any } \lambda \in [0,1]$$

In this case, $(x_1, x_2) \succsim (y_1, y_2)$ implies that $x_1 \geq y_1 - 1$ and $x_2 \leq y_2 + 1$; whereas $\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2) \succsim (y_1, y_2)$ implies

$$\begin{aligned} \lambda x_1 + (1-\lambda) y_1 &\geq y_1 - 1 \quad \text{for the first component, and} \\ \lambda x_2 + (1-\lambda) y_2 &\leq y_2 + 1 \quad \text{for the second component;} \end{aligned}$$

which respectively can be rewritten as

$$\begin{aligned} \lambda (x_1 - y_1) &\geq -1, \text{ and} \\ \lambda (x_2 - y_2) &\leq 1 \end{aligned}$$

In addition, the condition on the first component $x_1 - y_1 \geqslant -1$ (or alternatively, $x_1 \geqslant y_1 - 1$) holds by assumption since $(x_1, x_2) \succsim (y_1, y_2)$. Similarly, the condition on the second component $x_2 - y_2 \leqslant 1$ (or alternatively, $x_2 \leqslant y_2 + 1$) is also satisfied by $(x_1, x_2) \succsim (y_1, y_2)$. Hence, the preference relation satisfies convexity.