# Homework #3 (Due on September 21st, 2020)

1. Consider the following utility function with constant elasticity of substitution (CES): $u(x_1, x_2) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{\frac{1}{\rho}}$ where $\rho \neq 0$ and $\rho \leq 1$. Show that:

   (a) When $\rho = 1$, indifference curves are linear (goods 1 and 2 are perfect substitutes).

   - If $\rho = 1$, then $u(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$. Then, the utility function becomes linear (goods are perfect substitutes). The equation of an indifference curve can be obtained solving for $x_2$,

   $$x_2 = \frac{U}{\alpha_2} - \frac{\alpha_1}{\alpha_2} x_1$$

   For instance, for $U = 10$, the linear indifference curve originates at $\frac{10}{\alpha_2}$ and crosses the $x_1-$axis at $0 = \frac{10}{\alpha_2} - \frac{\alpha_1}{\alpha_2} x_1$, i.e., at $x_1 = \frac{10}{\alpha_1}$.

   (b) When $\rho \to 0$, the utility function represents a Cobb-Douglas utility function, $u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$, where the exponents satisfy $\alpha_1 + \alpha_2 = 1$.

   - Let us define $\widetilde{u}(x) \equiv \ln u(x)$, where

   $$\ln u(x) = \frac{1}{\rho} \ln \left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right].$$

   Then, the limit of this utility function when $\rho \to 0$ is

   $$\lim_{\rho \to 0} \widetilde{u}(x) = \lim_{\rho \to 0} \frac{\ln \left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]}{\rho} = \frac{0}{0}$$

   Hence, we need to use l'Hopital's rule, as follows,

   $$\lim_{\rho \to 0} \frac{\frac{\partial \ln\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]}{\partial \rho}}{\frac{\partial \rho}{\partial \rho}} = \lim_{\rho \to 0} \frac{\partial \ln \left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]}{\partial \rho} =$$

   $$\lim_{\rho \to 0} \frac{1}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \left[\alpha_1 \ln(x_1) x_1^\rho + \alpha_2 \ln(x_2) x_2^\rho\right] =$$

   $$\frac{\alpha_1 \ln(x_1) + \alpha_2 \ln(x_2)}{\alpha_1 + \alpha_2} = \frac{\left[x_1^{\alpha_1} x_2^{\alpha_2}\right]}{\alpha_1 + \alpha_2}$$

   Recall that so far we have been dealing with the limit $\lim_{\rho \to 0} \widetilde{u}(x)$, while in fact we are interested in $\lim_{\rho \to 0} u(x)$. So, given that $\widetilde{u}(x) \equiv \ln u(x)$, we have that $\lim_{\rho \to 0} \widetilde{u}(x) = x_1^{\alpha_1} x_2^{\alpha_2}$. Solving for $x_2$ yields the equation of an indifference curve $x_2 = \left(\frac{\widetilde{u}}{x_1^{\alpha_1}}\right)^{\frac{1}{\alpha_2}}$, e.g., if $\alpha_1 = \alpha_2 = 0.5$, the indifference curve becomes $x_2 = \frac{\widetilde{u}}{x_1}$.

Hence, indifference curves are decreasing in $x_1$, but at a decreasing rate (i.e., indifference curves are bowed-in towards the origin).

(c) When $\rho \to -\infty$, the utility function becomes a Leontief utility function given by $u(x_1, x_2) = \min\{x_1, x_2\}$, and thus represents two goods that are perfect complements. [*Hint*: Since in this case $\rho \to -\infty$, you can consider that $\rho$ is a negative number.]

- We need to show that if $x_2 \geq x_1$, then

$$\lim_{\rho \to -\infty} \left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{\rho}} = \min\{x_1, x_2\} = x_1.$$

- Suppose, without loss of generality, that $x_2 \geq x_1$. Then $x_2^\rho \leq x_1^\rho$ since parameter $\rho$ satisfies $0 \neq \rho \leq 1$ and approaches $-\infty$ (i.e., it is a negative number). Hence, multiplying both sides of the inequality by $\alpha_2 \geq 0$ yields $\alpha_2 x_2^\rho \leq \alpha_2 x_1^\rho$. Adding $\alpha_1 x_1^\rho$ on both sides of the inequality,

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho \leq \alpha_1 x_1^\rho + \alpha_2 x_1^\rho$$

and rearranging,

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho \leq (\alpha_1 + \alpha_2)\, x_1^\rho$$

and since $\rho$ is a negative number,

$$\underbrace{\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{\rho}}}_{B} \;\geq\; \underbrace{\left[(\alpha_1 + \alpha_2)\, x_1^\rho\right]^{\frac{1}{\rho}}}_{C} \tag{1}$$

- In addition, if $x_1 \geq 0$ and $x_2 \geq 0$, then $0 \leqslant \alpha_2 x_2^\rho$. If we add $\alpha_1 x_1^\rho$ on both sides of the inequality, we obtain

$$\alpha_1 x_1^\rho \leq \alpha_1 x_1^\rho + \alpha_2 x_2^\rho$$

and since $\rho$ is a negative number,

$$\underbrace{\left[\alpha_1 x_1^\rho\right]^{\frac{1}{\rho}}}_{A} \;\geq\; \underbrace{\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{\rho}}}_{B}$$

Combining this result with that of expression (1), yields

$$\underbrace{\left[\alpha_1 x_1^\rho\right]^{\frac{1}{\rho}}}_{A} \;\geq\; \underbrace{\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{\rho}}}_{B} \;\geq\; \underbrace{\left[(\alpha_1 + \alpha_2)\, x_1^\rho\right]^{\frac{1}{\rho}}}_{C} \tag{2}$$

We can use the "Squeezing Theorem" to obtain the limits of terms A and C, where $\rho \to -\infty$, in order to obtain the limit of term B, which must be between A and C. Let us first find the limit of term A,

$$\lim_{\rho \to -\infty} [\alpha_1 x_1^{\rho}]^{\frac{1}{\rho}} = x_1$$

Let us now find the limit of term C,

$$\lim_{\rho \to -\infty} [(\alpha_1 + \alpha_2)\, x_1^{\rho}]^{\frac{1}{\rho}} = x_1$$

Hence, since the limits of both terms A and C coincide (both of them are $x_1$), the limit of term B must also be $x_1$. That is,

$$\lim_{\rho \to -\infty} [\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}]^{\frac{1}{\rho}} = x_1$$

which is exactly what we needed to show: if $x_2 \geq x_1$, then

$$\lim_{\rho \to -\infty} [\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}]^{\frac{1}{\rho}} = \min\{x_1, x_2\} = x_1.$$

A similar argument applies to the case in which $x_2 < x_1$, whereby

$$\lim_{\rho \to -\infty} [\alpha_1 x_1^{\rho} + \alpha_2 x_2]^{\frac{1}{\rho}} = \min\{x_1, x_2\} = x_2.$$

1. Consider an individual facing price vector $p = (p_1, p_2) >> 0$ and income $w > 0$. If, after solving his UMP, his indirect utility function is $v(p, w) = \left(p_1^{\alpha} p_2^{1-\alpha}\right) w$, show that his utility function $u(x)$ must have a Cobb-Douglas representation, where $x = (x_1, x_2)$.

   - *Proof (EMP approach).* First, recall from duality that

   $$u(x) \equiv \min_{p \in \mathbb{R}_+^2} v(p, p \cdot x) \tag{1}$$

   Since $v(p, w)$ is homogeneous of degree zero, we can divide both arguments by $p \cdot x$ to obtain that the indirect utility function $v(p, w)$ is unaffected, i.e., $v(p, w) = v\left(\frac{p}{p \cdot x}, 1\right)$. Let $\overline{p} \equiv \frac{p}{p \cdot x}$, and thus $v(p, w) = v(\overline{p}, 1)$. As a consequence, if price vector $p^*$ minimizes $v(p, p \cdot x)$ for an income level $p \cdot x = w$, then price vector $\overline{p}$ minimizes $v(p, 1)$ for an income level $p \cdot x = 1$. That is, we can rewrite program (1) as

   $$u(x) \equiv \min_{p \in \mathbb{R}_+^2} v(p, 1) \quad \text{subject to } p \cdot x = 1 \tag{2}$$

- We can now find the price vector $p = (p_1, p_2)$ that solves program (2). Plugging them afterwards in the indirect utility function $v(p, 1)$ will yield the original utility function $u(x)$ that this consumer maximized in his UMP (as stated in (2)). Since program (2) is a constrained minimization problem, we set up the Lagrangian

$$\mathcal{L} = \left(p_1^{\alpha} p_2^{1-\alpha}\right) - \lambda(p_1 x_1 + p_2 x_2 - 1)$$

  Taking first-order conditions with respect to $p_1$ and $p_2$ yields, respectively

$$\begin{aligned} \frac{d\mathcal{L}}{dp_1} &= \alpha p_1^{\alpha-1} p_2^{1-\alpha} - \lambda x_1 = 0 \\ \frac{d\mathcal{L}}{dp_2} &= (1-\alpha) p_1^{\alpha} p_2^{-\alpha} - \lambda x_2 = 0 \end{aligned}$$

  and

$$\frac{d\mathcal{L}}{d\lambda} = -p_1 x_1 - p_2 x_2 + 1 = 0$$

  and simultaneously solving for $p_1$ and $p_2$ we obtain

$$p_1^* = \frac{\alpha}{x_1} \text{ and } p_2^* = \frac{1-\alpha}{x_2}$$

  We can finally plug these two prices, which solve (2), into the indirect utility funciton $v(p, 1)$, yielding

$$v(p_1^*, p_2^*, 1) = \left(\left(\frac{\alpha}{x_1}\right)^{\alpha} \left(\frac{1-\alpha}{x_2}\right)^{1-\alpha}\right)(1) = \underbrace{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}_{\text{constant}} x_1^{-\alpha} x_2^{\alpha-1}$$

  which is clearly of the Cobb-Douglas type. For instance, labeling $A \equiv \alpha^{\alpha}(1-\alpha)^{1-\alpha}$ yields $v(p_1^*, p_2^*, 1) = A x_1^{-\alpha} x_2^{\alpha-1}$, thus taking a more familiar format.

2. Consider an individual with a separable utility function over $L$ goods

$$u(x) = \sum_{i=1}^{L} \alpha_i \ln x_i,$$

where $\sum_{i=1}^{L} \alpha_i = 1$ and $\alpha_i > 0$ for every good $i$. Assume that the consumer faces a strictly positive price vector $p \gg 0$ and his wealth is given by $w > 0$.

   (a) Find the Walrasian demands, and the shadow price of wealth.

- The consumer solves a UMP given by

$$\max_{x \geq 0} \quad u(x)$$

$$\text{subject to } p \cdot x \leq w$$

Using the shortcut $MRS_{i,j} = \frac{p_i}{p_j}$, we obtain interior solutions $\frac{\alpha_i}{x_i} = \frac{p_i}{p_j}$, or $\frac{\alpha_i}{p_i} \cdot p_j = x_i$, which together with the budget constraint yields a Walrasian demand of

$$x_i(p, w) = \frac{\alpha_i w}{p_i} \quad \text{for every good } i \tag{1}$$

In addition, we can obtain the Lagrange multiplier, $\lambda$, from the first-order condition

$$\frac{\partial u}{\partial x_i} = \lambda p_i, \text{ or } \frac{\alpha_i}{x_i} = \lambda p_i$$

which, combined with (1) yields

$$\frac{\alpha_i}{\frac{\alpha_i w}{p_i}} = \lambda p_i$$

and solving for $\lambda$ we obtain

$$\lambda(p, w) = \frac{1}{w}$$

Hence, the marginal value of relaxing the constraint (i.e., the shadow price of wealth) is $\frac{1}{w}$.

(b) Let us next find the shadow price of wealth using an alternative approach. First, find the indirect utility function, $v(p, w)$, resulting from the previous UMP. Then, measure how it is affected by a marginal increase in wealth, i.e., find the derivative $\frac{\partial v(p,w)}{\partial w}$. Does your result coincide with what you found in part (a)?

- The indirect utility function is

$$v(p, w) = \sum_{i=1}^{L} \alpha_i \cdot \ln \left( \frac{\alpha_i w}{p_i} \right)$$

Hence, the marginal utility of wealth is

$$\frac{\partial v(p, w)}{\partial w} = \sum_{i=1}^{L} \alpha_i \cdot \frac{1}{\frac{\alpha_i w}{p_i}} \cdot \frac{\alpha_i}{p_i} = \frac{1}{w} \sum_{i=1}^{L} \alpha_j = \frac{1}{w}$$

which coincides with the Lagrange multiplier $\lambda(p, w)$ we found in part (a).

- Interestingly, this result is generalizable to settings in which, given the separable nature of the utility function, the consumer focuses on a subset of goods $\{1, 2, ..., L_1\}$ where $L_1 < L$, $\{L_1+1, ..., L_2\}$, etc. and solves a separated UMP for each of these subsets of goods, i.e., one UMP for goods $\{1, 2, ..., L_1\}$, another UMP for goods $\{L_1+1, ..., L_2\}$, etc.. The consumer's solution to these separated UMPs must coincide with that in part (a), where the consumer simultaneously considers all $L$ goods.

3. Consider the utility function in question (1) and assume $\alpha_1 = \alpha_2 = 1$. Derive its Hicksian demand function and expenditure function. In addition, verify that the Hicksian demand satisfies: (i) homogeneity of degree zero in prices, (ii) no excess of utility, and (iii) convexity; and that the expenditure function satisfies: (i) homogeneity of degree one in prices, (ii) strictly increasing in utility and nondecreasing in prices, (iii) concave in prices and (iv) continuous in prices and utility.

- Derive its Hecksian demand function and expenditure function.

  The utility function is

$$u(x) = \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}}$$

  Differentiating the utility function with respect to $x_1$,

$$u_{x_1} = x_1^{\rho-1}\left(x_1^\rho + x_2^\rho\right)^{\frac{1-\rho}{\rho}}$$

  Differentiating the utility function with respect to $x_2$,

$$u_{x_2} = x_2^{\rho-1}\left(x_1^\rho + x_2^\rho\right)^{\frac{1-\rho}{\rho}}$$

  For utility maximization to hold,

$$\frac{p_1}{p_2} = \frac{u_{x_1}}{u_{x_2}}$$

  such that

$$\frac{p_1}{p_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1}$$

  Therefore,

$$x_1 = x_2\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} \quad \text{, and}$$

$$x_2 = x_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}}$$

Substituting into the utility function,

$$u = p_1^{-\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}} x_1 \quad \text{, and}$$
$$u = p_2^{-\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}} x_2$$

such that the Hecksian demand correspondence becomes

$$h_1(p,u) = p_1^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} u \quad \text{, and}$$
$$h_2(p,u) = p_2^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} u$$

Therefore, the expenditure function becomes

$$\begin{aligned} e(p,u) &= p \cdot h(p,u) \\ &= p_1 \cdot h_1(p,u) + p_2 \cdot h_2(p,u) \\ &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} u \end{aligned}$$

- To verify Proposition 3.E.2, the expenditure function, $e(p,u)$, should satisfy

[(i)] Homogeneous of degree one in $p$

- Check that

$$\begin{aligned} e(\lambda p,u) &= \left((\lambda p_1)^{\frac{\rho}{\rho-1}} + (\lambda p_2)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} u \\ &= (\lambda)^{\frac{\rho}{\rho-1}\frac{\rho-1}{\rho}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} u \\ &= \lambda e(p,u) \end{aligned}$$

[ii] Strictly increasing in $u$ and non-decreasing in $p$

$$\begin{aligned} e_u(p,u) &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} > 0 \\ e_{p_1}(p,u) &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} p_1^{\frac{1}{\rho-1}} \geq 0 \\ e_{p_2}(p,u) &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} p_2^{\frac{1}{\rho-1}} \geq 0 \end{aligned}$$

[iii] Concave in $p$

- The Hessian matrix of the expenditure function is

$$\begin{pmatrix} e_{p_{11}} & e_{p_{12}} \\ e_{p_{21}} & e_{p_{22}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho-1} p_1^{\frac{2-\rho}{\rho-1}} p_2^{\frac{\rho}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} & -\frac{1}{\rho-1} (p_1 p_2)^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} \\ -\frac{1}{\rho-1} (p_1 p_2)^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} & \frac{1}{\rho-1} p_1^{\frac{\rho}{\rho-1}} p_2^{\frac{2-\rho}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} \end{pmatrix}$$

for which the determinant is given by

$$\Delta_e = \left(\frac{1}{\rho-1}\right)^2 \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{2(1+\rho)}{\rho}} \left(p_1^{\frac{2-\rho}{\rho-1}+\frac{\rho}{\rho-1}} p_2^{\frac{\rho}{\rho-1}+\frac{2-\rho}{\rho-1}} - (p_1 p_2)^{\frac{2}{\rho-1}}\right) = 0$$

such that the expenditure function satisfies concavity.

[iv] Continuous in $p$ and $u$

Since $e(p,u)$ is differentiable in $p$ and in $u$, it is continuous in the respective arguments.

- To verify Proposition 3.E.3, the Hicksian demand correspondence, $h(p,u)$, should satisfy

[(i)] Homogeneity of degree zero in $p$

$$\begin{aligned} h_1(\lambda p, u) &= (\lambda p_1)^{\frac{1}{\rho-1}} \left((\lambda p_1)^{\frac{\rho}{\rho-1}} + (\lambda p_2)^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} u \\ &= \lambda^{\left(\frac{1}{\rho-1}-\frac{1}{\rho-1}\right)} p_1^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} u \\ &= h_1(p,u) \end{aligned}$$

and analogously for $h_2(\lambda p, u) = h_2(p,u)$.

[ii] No excess utility

$$\begin{aligned} u(h(p,u)) &= \left([h_1(p,u)]^\rho + [h_2(p,u)]^\rho\right)^{\frac{1}{\rho}} \\ &= \left(\left[p_1^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} u\right]^\rho + \left[p_2^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} u\right]^\rho\right)^{\frac{1}{\rho}} \\ &= u \left(p_1^{\frac{\rho}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-1} + p_2^{\frac{\rho}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-1}\right)^{\frac{1}{\rho}} \\ &= u \end{aligned}$$

[iii] Convexity/ Uniqueness

Consider the Hessian matrix of $u(x)$,

$$\begin{pmatrix} u_{x_1 1} & u_{x_1 2} \\ u_{x_2 1} & u_{x_2 2} \end{pmatrix} = \begin{pmatrix} (1-\rho)\, x_1^{\rho-2} x_2^\rho \left(x_1^\rho + x_2^\rho\right)^{\frac{1-2\rho}{\rho}} & (1-\rho)(x_1 x_2)^{\rho-1} \left(x_1^\rho + x_2^\rho\right)^{\frac{1-2\rho}{\rho}} \\ (1-\rho)(x_1 x_2)^{\rho-1} \left(x_1^\rho + x_2^\rho\right)^{\frac{1-2\rho}{\rho}} & (1-\rho)\, x_1^\rho x_2^{\rho-2} \left(x_1^\rho + x_2^\rho\right)^{\frac{1-2\rho}{\rho}} \end{pmatrix}$$

for which the determinant is given by

$$\Delta_u = (1-\rho)^2 \left(x_1^\rho + x_2^\rho\right)^{\frac{2(1-2\rho)}{\rho}} (x_1 x_2)^{2\rho-2-2(\rho-1)} = 0$$

By Theorem M.C.4 (p.935), we have semi-definiteness but not strict quasiconcavity of $u(x)$. Therefore, we do not have a sufficient condition to establish the uniqueness of the Hicksian demand correspondence $h(p,u)$. For instance, when $\rho \to -\infty$, $u(x) = \min\{x_1, x_2\}$ such that the Leontiff utility function (in part (c) of problem 3.C.6) is not strictly quasiconcave (for the underlying preference relation, $\succsim$, is not strictly convex) and does not possess a unique element in $h(p,u)$.