Homework #3 (Due on September 21st, 2020)

1. Consider the following utility function with constant elasticity of substitution (CES): $u(x_1, x_2) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{\frac{1}{\rho}}$ where $\rho \neq 0$ and $\rho \leq 1$. Show that:

   (a) When $\rho = 1$, indifference curves are linear (goods 1 and 2 are perfect substitutes).

   (b) When $\rho \to 0$, the utility function represents a Cobb-Douglas utility function, $u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$, where the exponents satisfy $\alpha_1 + \alpha_2 = 1$.

   (c) When $\rho \to -\infty$, the utility function becomes a Leontief utility function given by $u(x_1, x_2) = \min\{x_1, x_2\}$, and thus represents two goods that are perfect complements. [*Hint*: Since in this case $\rho \to -\infty$, you can consider that ρ is a negative number.]

2. Consider an individual facing price vector $p = (p_1, p_2) >> 0$ and income $w > 0$. If, after solving her UMP, her indirect utility function is $v(p, w) = \left(p_1^\alpha p_2^{1-\alpha}\right) w$, show that her utility function $u(x)$ must have a Cobb-Douglas representation, where $x = (x_1, x_2)$.

3. Consider an individual with a separable utility function over L goods

$$u(x) = \sum_{i=1}^{L} \alpha_i \ln x_i,$$

   where $\sum_{i=1}^{L} \alpha_i = 1$ and $\alpha_i > 0$ for every good i. Assume that the consumer faces a strictly positive price vector $p \gg 0$ and his wealth is given by $w > 0$.

   (a) Find the Walrasian demands, and the shadow price of wealth.

   (b) Let us next find the shadow price of wealth using an alternative approach. First, find the indirect utility function, $v(p, w)$, resulting from the previous UMP. Then, measure how it is affected by a marginal increase in wealth, i.e., find the derivative $\frac{\partial v(p,w)}{\partial w}$. Does your result coincide with what you found in part (a)?

4. Consider the utility function in question (1) and assume $\alpha_1 = \alpha_2 = 1$. Derive its Hicksian demand function and expenditure function. In addition, verify that the Hicksian demand satisfies: (i) homogeneity of degree zero in prices, (ii) no excess of utility, and (iii) convexity; and that the expenditure function satisfies: (i) homogeneity of degree one in prices, (ii) strictly increasing in utility and nondecreasing in prices, (iii) concave in prices and (iv) continuous in prices and utility.