Homework # 2 EconS501 [Due on Sepetember 14th, 2020]

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1. Consider a K-commodity setting in which a consumer's Walrasian demand function is

$$x_j(p, w) = \frac{w}{\sum_{k=1}^{K} p_k} \text{ for } j = 1, ...K$$

   (a) Show that this demand function is homogeneous of degree zero in (p, w).

Consider that

$$\begin{aligned} x_j(\alpha p, \alpha w) &= \frac{\alpha w}{\sum_{k=1}^{K} \alpha p_k} \\ &= \frac{\alpha w}{\alpha \sum_{k=1}^{K} p_k} \\ &= x_j(p, w) \end{aligned}$$

1. (b) Show that it satisfies Walra's Law.

Consider that

$$\begin{aligned} p \times x_j(p, w) &= \sum_j p_j \times x_j(p, w) \\ &= \sum_j p_j \times \frac{w}{\sum_{k=1}^{K} p_k} \\ &= w \end{aligned}$$

1. (c) Show that the weak axiom is satisfied.

Suppose that

$$p' \times x(p, w) \leq w' \text{ and } p \times x(p', w') \leq w$$

The first inequality implies that

$$\sum_k p'_k \times \frac{w}{\sum_k p_k} \leq w' \text{ that is } \frac{w}{\sum_k p_k} \leq \frac{w'}{\sum_k p'_k}$$

The second inequality implies that

$$\sum_k p_k \times \frac{w'}{\sum_k p'_k} \leq w \text{ that is } \frac{w'}{\sum_k p'_k} \leq \frac{w}{\sum_k p_k}$$

Therefore,

$$\frac{w}{\sum_k p_k} = \frac{w'}{\sum_k p'_k}$$

and, hence, $x(p, w) = x(p', w')$ implying that the weak axiom is satisfied.

1. (d) Identify the Slutsky substitution matrix for this demand function. Show that it is symmetric and negative semidefinite.

By calculation, we obtain

$$D_w x(p, w) = \left(\frac{1}{\sum_k p_k}\right) \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

$$D_p x(p, w) = \left(-\frac{w}{\left(\sum_k p_k\right)^2}\right) \begin{bmatrix} 1 & . & . & 1 \\ . & & & \\ . & & & \\ 1 & . & . & 1 \end{bmatrix}$$

Hence, $S(p, w) = 0$ is symmetric, negative semidefinite but not negative definite.

2. Consider a consumer with utility function $u(x_1, x_2, x_3)$ for three goods, where the cross-price elasticities are null. Show that the ratio of substitution effects $\frac{s_{23}}{s_{13}}$ is equal to $\frac{\frac{\partial x_2}{\partial w}}{\frac{\partial x_1}{\partial w}}$.

- First, note that the substitution effect $s_{23} = \frac{\partial h_2(p,u^0)}{\partial p_3}$ can be found from the Slutsky equation,

$$\underbrace{\frac{\partial h_2(p, u^0)}{\partial p_3}}_{SE} = \underbrace{\frac{\partial x_2(p, w)}{\partial p_3}}_{TE} + \underbrace{\frac{\partial x_2(p, w)}{\partial w} x_3(p, w)}_{IE}$$

  where the first term on the right-hand side of the equality, $\frac{\partial x_2(p,w)}{\partial p_3}$, measures the total effect of a marginal change in p_3, while the second term, $\frac{\partial x_2(p,w)}{\partial w} x_3(p, w)$, measures the income effect. In addition, we know that cross-price elasticity are null, i.e., $\varepsilon_{23} = 0$, implying that an increase in p_3 does not affect the Walrasian demand of good 2, i.e., $\frac{\partial x_2(p,w)}{\partial p_3} = 0$, entailing that the total (cross-price) effect is null. Hence, the above expression of the Slutsky equation reduces to

$$\frac{\partial h_2(p,u^0)}{\partial p_3} = \frac{\partial x_2(p,w)}{\partial w} x_3(p,w)$$

Similarly, substitution effect $s_{13} = \frac{\partial h_1(p,u^0)}{\partial p_3}$ can be found by using the Slutsky equation,

$$\frac{\partial h_1(p,u^0)}{\partial p_3} = \frac{\partial x_1(p,w)}{\partial p_3} + \frac{\partial x_1(p,w)}{\partial w} x_3(p,w) = \frac{\partial x_1(p,w)}{\partial w} x_3(p,w)$$

where the second equality also uses the property that cross-price elasticities are null, i.e., $\varepsilon_{13} = 0$ and thus $\frac{\partial x_1(p,w)}{\partial p_3} = 0$. Hence, the ratio of substitution effects $\frac{s_{23}}{s_{13}}$ reduces to

$$\frac{s_{23}}{s_{13}} = \frac{\frac{\partial x_2(p,w)}{\partial w} x_3(p,w)}{\frac{\partial x_1(p,w)}{\partial w} x_3(p,w)} = \frac{\frac{\partial x_2(p,w)}{\partial w}}{\frac{\partial x_1(p,w)}{\partial w}}.$$

3. Consider an individual with the following utility function for goods x and y, and for the numeraire (good z, whose price is normalized to \$1),

$$u(x,y,z) = a\,(x+y) - \frac{1}{2}\left[b(x^2+y^2) + 2dxy\right] + z$$

where $b > |d|$. This assumption implies that the consumer regards goods x and y as differentiated, and we return to this property when solving the exercise. Consumer's budget constraint is $w = z + p_x x + p_y y$, where $p_x, p_y > 0$.

(a) Solve this consumer's utility maximization problem to find his Walrasian demand for goods x and y. Show that Walrasian demands are linear in the price of good x and y.

- The consumer solves

$$\max_{x,y,z} u(x,y,z) = a\,(x+y) - \frac{1}{2}\left[b\left(x^2+y^2\right) + 2dxy\right] + z$$

$$\text{subject to } \; w = z + p_x x + p_y y$$

The tangency conditions in this setting are

$$\begin{aligned} \frac{MU_x}{MU_z} &= \frac{p_x}{p_z} \iff \frac{a - bx - dy}{1} = p_x \\ \frac{MU_y}{MU_z} &= \frac{p_y}{1} \iff \frac{a - dx - by}{1} = p_x \end{aligned}$$

Solving for p_x in the first tangency condition, we obtain

$$p_x = a - bx - dy$$

and solving for p_y in the second tangency condition yields

$$p_y = a - by - dx.$$

These are the inverse demand function, describing the consumer's maximum willingness to pay for goods x and y.

- Solving for x in $p_x = a - bx - dy$, yields

$$x = \frac{a}{b} - d\frac{y}{b} - \frac{1}{b}p_x$$

and solving for y in $p_y = a - by - dx$, we find

$$y = \frac{a}{b} - d\frac{x}{b} - \frac{1}{b}p_y.$$

Inserting one expression into the other, yields

$$x = \frac{a}{b} - d\frac{\overbrace{\frac{a}{b} - d\frac{x}{b} - \frac{1}{b}p_y}^{y}}{b} - \frac{1}{b}p_x$$

Rearranging this equation and solving for x, we find the Walrasian demand for good x,

$$x(p,w) = \frac{a}{b+d} - \frac{b}{b^2-d^2}p_x + \frac{d}{b^2-d^2}p_y$$

Therefore, the Walrasian demand for good y is

$$\begin{aligned} y(p,w) &= \frac{a}{b} - d\frac{\overbrace{\frac{a}{b+d} - \frac{b}{b^2-d^2}p_x + \frac{d}{b^2-d^2}p_y}^{x(p,w)}}{b} - \frac{1}{b}p_y \\ &= \frac{a}{b+d} + \frac{d}{b^2-d^2}p_x - \frac{b}{b^2-d^2}p_y \end{aligned}$$

The above Walrasian demands can be more compactly presented as

$$\begin{aligned} x(p,w) &= \widetilde{a} - \widetilde{b}p_x + \widetilde{d}p_y, \text{ and} \\ y(p,w) &= \widetilde{a} + \widetilde{d}p_x - \widetilde{b}p_y \end{aligned}$$

where $\widetilde{a} \equiv \frac{a}{b+d}$, $\widetilde{b} \equiv \frac{b}{b^2-d^2}$, and $\widetilde{d} \equiv \frac{d}{b^2-d^2}$.

(b) Interpret your results in terms of parameters b and d.

- The exercise assumes that parameter b satisfies $b > |d|$. This assumption entails that $\widetilde{b} > \left|\widetilde{d}\right|$ which, in terms of the above Walrasian demands, means that the consumer purchases of good x are more sensitive to the price of this good, p_x, than to the price of good y, p_y. This assumption is more compactly described as that, in the demand for good x, "own-price" effects dominate "cross-price" effects. A similar argument applies to the Walrasian demand of good y, which is more affected by a change in its own price, p_y, than in the other good's price, p_x.

(c) Evaluate the Walrasian demands in the case that $d = 0$. Interpret.

- In that setting, the Walrasian demands we found in part (b) simplify to

$$\begin{aligned} x(p,w) &= \frac{a}{b} - \frac{b}{b^2}p_x \\ &= \frac{a}{b} - \frac{1}{b}p_x \end{aligned}$$

which coincides with the standard Walrasian demand associated to a linear inverse demand $p_x(x) = a - bx$, i.e., solving for x, we find $x(p,w) = \frac{a}{b} - \frac{1}{b}p_x$. Intuitively, the demand for good x only depends on its own price, p_x.

- A similar argument applies for the Walrasian demand of good y,

$$\begin{aligned} y(p,w) &= \frac{a}{b} - \frac{b}{b^2}p_y \\ &= \frac{a}{b} - \frac{1}{b}p_y \end{aligned}$$

which is a function of price p_y alone.

4. Consider utility function $u(x,y)$, where x and y represent the units of two goods. Assume that $u(\cdot)$ is twice continuously differentiable, strictly increasing and concave in both of its arguments, x and y. Assuming that the consumer's wealth is given by $w > 0$, and that he faces a price vector $p = (p_x, p_y) >> 0$, denote his indirect utility function as $v(p,w)$.

(a) Use the indirect utility function $v(p,w)$ to find the consumer willingness to pay for good y.

- The indirect utility function can be found by solving the consumer's utility maximization problem subject to her budget constraint as follows:

$$v(p, w, y) = \max u(x, y) \quad \text{s.t.} \quad p_x x + p_y y \leq w.$$

Define the marginal rate of substitution between income and good y, $MRS_{y,w}$, such that:

$$MRS_{y,w} = \frac{v_y}{v_w}$$

where $v_y = \frac{\partial v}{\partial y}$ and $v_w = \frac{\partial v}{\partial w}$. Then, define the willingness to pay for good y as the product $WTP = MRS_{y,w} \times y$.

(b) Identify under which condition is this willingness to pay for good y increasing or decreasing in income, w. Interpret.

- To examine how WTP for good y varies with income, w, we need to determine the income effect $\frac{\partial WTP}{\partial w}$. It may be helpful to estimate the value of the income elasticity of WTP, which is defined as:

$$\varepsilon_{WTP}^{w} = \frac{\frac{\partial WTP}{WTP}}{\frac{\partial w}{w}} = \frac{\partial WTP}{\partial w} \frac{w}{WTP}$$

Since $w > 0$, $y > 0$ and $WTP > 0$, we obtain that and $\frac{w}{WTP} > 1$. Therefore, ε_{WTP}^{w} has the same sign as $\frac{\partial WTP}{\partial w}$. Since $WTP = MRS_{y,w} \times y$ by definition, $\frac{\partial WTP}{\partial w}$ has the same sign as $\frac{\partial MRS_{y,w}}{\partial w}$. Let us next find this derivative

$$\frac{\partial MRS_{y,w}}{\partial w} = \frac{v_{wy} v_w - v_{ww} v_y}{v_w^2}$$

where $v_w > 0$, $v_y > 0$, and by assumption $v_{ww} < 0$. Hence, the sign of $\frac{\partial MRS_{y,w}}{\partial w}$ depends on the sign of the cross derivative v_{wy}, which intuitively indicates the interaction between income and good y in the utility function. Hence, we can identify two cases:

  - $\frac{\partial WTP}{\partial w} < 0$, implying that the willingness to pay for good y decreases with income, only if income and good y are regarded as substitutes or independent by the consumer, i.e., $v_{wy} < 0$ or $v_{wy} = 0$.
  - The opposite case, $\frac{\partial WTP}{\partial w} > 0$, indicating that the willingness to pay for good y increases with income, can occur: (1) under complementarity (i.e., $v_{wy} > 0$); and (2) under substitutability ($v_{wy} < 0$ if, in addition, the numerator of $\frac{\partial MRS_{y,w}}{\partial w}$ is negative, that is $|v_{wy} v_w| < |v_{ww} v_y|$).