Homework #7 (Due on October 21st, 2019)

1. Consider an agent who exerts an effort level e, where $e \geq 0$, to generate output y that is subject to output shocks ε (e.g., weather conditions affecting the quality of harvest, machine breakdown causing product failure, etc.). Output then behaves as follows

$$y = ge + \varepsilon$$

where g denotes the agent's output efficiency, and shock ε follows a normal distribution, $N(0, \sigma^2)$, with mean of zero and variance $\sigma^2 > 0$. The agent earns a wage of $w = sy$, where $0 \leq s \leq 1$ represents his output shares (e.g., commission from sales of the products). In addition, the agent incurs a cost to exert effort e, given by

$$c(e) = \frac{1}{2}e^2$$

which is increasing and convex in his effort level e.

The agent's payoff comes from the utility of earning wage w minus his cost of effort, where

$$U = u(w) - c(e)$$

Specifically, his utility function follows the negative exponential form of

$$u(w) = 1 - \exp(-\eta w) \tag{1}$$

   (a) What is the agent's Arrow-Pratt coefficient of absolute risk aversion, $r_A(w)$. How does it vary with his wage?

   (b) Find the certainty equivalent of the agent.

   (c) Find the agent's optimal effort e^*. How does it vary with his output share s, output efficiency g, risk aversion η, and output shocks σ?

2. Consider an individual with the following utility function, where x denotes income.

$$u(x) = \begin{cases} 2x & \text{if} \quad x \leq \frac{5}{2} \\ \frac{5}{2} + x & \text{if} \quad x > \frac{5}{2} \end{cases}$$

   (a) Depict the utility function with $u(x)$ on the vertical axis and income, x, on the horizontal axis. Show that this individual is (weakly) risk averse.

   (b) Suppose that there are three states of the world, each equally likely. There are two assets, x and y. The asset x is the random variable with payoffs $(1, 5, 9)$

and the asset y is the random variable with payoffs $(2, 3, 10)$. (Note that assets specify a payoff triple, to indicate the payoff arising in each of the three equally likely states of the world.) Calculate the expected utility of asset x and of asset y. Which asset, hence, would be preferred by this individual, if both of them were offered at the same price?

(c) Calculate the expected *value* of each asset (you previously found the expected *utility*). Calculate the variance of both assets. Which asset would be chosen by this individual if he were variance averse?

(d) From your previous answers, comment on the validity of the following statement: "Every risk-averse individual is also variance averse".

3. Let G be the set of compound gambles over a finite set of deterministic payoffs $\{a_1, a_2, ...a_n\} \subset \mathbb{R}_+$. A decision maker's preference relation $\succsim$ over compound gambles can be represented by utility function $v : G \rightarrow \mathbb{R}$. Let $g \in G$, and let probability p_i be associated to the corresponding payoff a_i. Finally, consider that the decision maker's utility function $v(\cdot)$ is given by

$$v(g) = (1 + a_1)^{p_1} (1 + a_1)^{p_2} ...(1 + a_n)^{p_n} = \prod_{i=1}^{n} (1 + a_i)^{p_i}$$

(a) Show that this is *not* a von Neumann-Morgenstern (vNM) utility function.

(b) Show that the decision maker has the same preference relation as an expected utility maximizer with von-Neumann Morgenstern utility function

$$u(g) = \sum_{i=1}^{n} p_i \ln (1 + a_i) .$$

(c) Assume now that the decision maker you considered in part (b) has utility function $u(w) = \ln (1 + w)$ over wealth $w \geqslant 0$. Evaluate his risk attitude (concavity in his utility function). Additionally, find the Arrow-Pratt coefficient of absolute risk aversion, $r_A(w, u)$. How does $r_A(w, u)$ change in wealth?