Recitation #3 - September 7th, 2019

1. Consider the following utility function with constant elasticity of substitution (CES): $u(x_1, x_2) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{\frac{1}{\rho}}$ where $\rho \neq 0$ and $\rho \leq 1$. Show that:

   (a) When $\rho = 1$, indifference curves are linear (goods 1 and 2 are perfect substitutes).

   - If $\rho = 1$, then $u(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$. Then, the utility function becomes linear (goods are perfect substitutes). The equation of an indifference curve can be obtained solving for x_2,

   $$x_2 = \frac{U}{\alpha_2} - \frac{\alpha_1}{\alpha_2} x_1$$

   For instance, for $U = 10$, the linear indifference curve originates at $\frac{10}{\alpha_2}$ and crosses the x_1-axis at $0 = \frac{10}{\alpha_2} - \frac{\alpha_1}{\alpha_2} x_1$, i.e., at $x_1 = \frac{10}{\alpha_1}$.

   (b) When $\rho \to 0$, the utility function represents a Cobb-Douglas utility function, $u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$, where the exponents satisfy $\alpha_1 + \alpha_2 = 1$.

   - Let us define $\widetilde{u}(x) \equiv \ln u(x)$, where

   $$\ln u(x) = \frac{1}{\rho} \ln\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right].$$

   Then, the limit of this utility function when $\rho \to 0$ is

   $$\lim_{\rho \to 0} \widetilde{u}(x) = \lim_{\rho \to 0} \frac{\ln\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]}{\rho} = \frac{0}{0}$$

   Hence, we need to use l'Hopital's rule, as follows,

   $$\lim_{\rho \to 0} \frac{\frac{\partial \ln\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]}{\partial \rho}}{\frac{\partial \rho}{\partial \rho}} = \lim_{\rho \to 0} \frac{\partial \ln\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]}{\partial \rho} =$$

   $$\lim_{\rho \to 0} \frac{1}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \left[\alpha_1 \ln(x_1) x_1^\rho + \alpha_2 \ln(x_2) x_2^\rho\right] =$$

   $$\frac{\alpha_1 \ln(x_1) + \alpha_2 \ln(x_2)}{\alpha_1 + \alpha_2} = \frac{\left[x_1^{\alpha_1} x_2^{\alpha_2}\right]}{\alpha_1 + \alpha_2}$$

   Recall that so far we have been dealing with the limit $\lim_{\rho \to 0} \widetilde{u}(x)$, while in fact we are interested in $\lim_{\rho \to 0} u(x)$. So, given that $\widetilde{u}(x) \equiv \ln u(x)$, we have that $\lim_{\rho \to 0} \widetilde{u}(x) = x_1^{\alpha_1} x_2^{\alpha_2}$. Solving for x_2 yields the equation of an indifference curve $x_2 = \left(\frac{\widetilde{u}}{x_1^{\alpha_1}}\right)^{\frac{1}{\alpha_2}}$, e.g., if $\alpha_1 = \alpha_2 = 0.5$, the indifference curve becomes $x_2 = \frac{\widetilde{u}}{x_1}$.

Hence, indifference curves are decreasing in x_1, but at a decreasing rate (i.e., indifference curves are bowed-in towards the origin).

2. Consider a setting with three goods ($L = 3$) and a consumer with Walrasian demand function $x(p, w)$ given by

$$x_1(p, w) = \frac{p_2}{p_3}; \;\; x_2(p, w) = -\frac{p_1}{p_3}; \;\; \text{and} \;\; x_3(p, w) = \frac{w}{p_3}$$

   (a) Show that the Walrasian demand is homogeneous of degree zero in prices and wealth, (p, w).

      - Increasing prices and wealth by a common factor λ, we obtain
         - $x_1(\lambda p, \lambda w) = \frac{\lambda p_2}{\lambda p_3} = \frac{p_2}{p_3} = x_1(p, w)$
         - $x_2(\lambda p, \lambda w) = -\frac{\lambda p_1}{\lambda p_3} = -\frac{p_1}{p_3} = x_2(p, w)$
         - $x_3(\lambda p, \lambda w) = \frac{\lambda w}{\lambda p_3} = \frac{w}{p_3} = x_3(p, w)$

         That is, increasing both prices and wealth by the same factor λ does not change this consumer's demand. Intuitively, if we double the price of all goods but also double his income, the individual's demand is unaffected.

   (b) Show that $x(p, w)$ satisfies Walras' law.

      - Recall that Walras' Law states that for a strictly positive price vector ($p >> 0$) and a positive wealth level ($w > 0$), $p \cdot x = w$, or alternatively, $\sum_{i=1}^{3} p_i x_i = w$. Hence, in this context,

$$\begin{aligned} \sum_{i=1}^{3} p_i x_i &= p_1 x_1(p, w) + p_2 x_2(p, w) + p_3 x_3(p, w) = \\ &= p_1 \frac{p_2}{p_3} + p_2 \left(-\frac{p_1}{p_3} \right) + p_3 \frac{w}{p_3} \end{aligned}$$

      and further rearranging, we obtain

$$\sum_{i=1}^{3} p_i x_i = \frac{p_1 p_2 - p_2 p_1 + p_3 w}{p_3} = w$$

      Therefore, Walras' Law is satisfied, confirming that the individual spends all his income on goods 1, 2 and 3.

   (c) Show that $x(p, w)$ violates the weak axiom of revealed preference (WARP).

- Let us use a counterexample.

$$\begin{array}{llll} w = 1 & p = (1,1,1) & \text{which yields a demand of} & x(p,w) = (1,-1,1) \\ w' = 2 & p' = (1,1,2) & \text{which yields a demand of} & x(p',w') = \left(\frac{1}{2}, \frac{-1}{2}, 1\right) \end{array}$$

We know that WARP is satisfied if for any pair of prices and wealth (p, w) and (p', w'),

$$p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \text{ then } p' \cdot x(p, w) > w'$$

In our example, the bundle that the consumer selects at the final price-wealth pair is affordable under initial prices and wealth,

$$p \cdot x(p', w') = [1, 1, 1] \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \frac{1}{2} - \frac{1}{2} + 1 = 1 \leq w \text{ (since } w = 1)$$

However, the consumption bundle at initial prices and wealth, $x(p, w)$, is affordable under final prices and wealth. In particular,

$$p' \cdot x(p, w) = [1, 1, 2] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1 - 1 + 2 = 2$$

Hence, since $w' = 2$, bundle $x(p, w)$ is exactly affordable at final prices and wealth, implying that the conclusion of WARP, $p' \cdot x(p, w) > w'$ is *not* satisfied. Therefore, WARP is violated.

(d) Find the Slutsky matrix $S(p, w)$.

- Let us first recall the Slutsky matrix:

$$S(p, w) = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1N} \\ s_{21} & s_{22} & \dots & s_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ s_{N1} & s_{N2} & \dots & s_{NN} \end{bmatrix}$$

where every component s_{ik} is defined as

$$s_{ik} = \frac{\partial x_i(p, w)}{\partial p_k} + \frac{\partial x_i(p, w)}{\partial w} x_k(p, w) \qquad \text{(Slutsky equation)}$$

The Slutsky equation informs us about the change in the demand for good i that results from a change in the price of good k once the consumer's wealth is appropriately compensated. Let us now find each of the components of the Slutsky matrix for this particular exercise.

$$\begin{aligned}
s_{11} &= \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w) = 0 + 0 = 0 \\
s_{12} &= \frac{\partial x_1(p,w)}{\partial p_2} + \frac{\partial x_1(p,w)}{\partial w} x_2(p,w) = \frac{1}{p_3} + 0 = \frac{1}{p_3} \\
s_{13} &= \frac{\partial x_1(p,w)}{\partial p_3} + \frac{\partial x_1(p,w)}{\partial w} x_3(p,w) = -\frac{p_2}{p_3^2} + 0 = -\frac{p_2}{p_3^2} \\
s_{21} &= \frac{\partial x_2(p,w)}{\partial p_1} + \frac{\partial x_2(p,w)}{\partial w} x_1(p,w) = -\frac{1}{p_3} + 0 = -\frac{1}{p_3} \\
s_{22} &= \frac{\partial x_2(p,w)}{\partial p_2} + \frac{\partial x_2(p,w)}{\partial w} x_2(p,w) = 0 + 0 = 0 \\
s_{23} &= \frac{\partial x_2(p,w)}{\partial p_3} + \frac{\partial x_2(p,w)}{\partial w} x_3(p,w) = \frac{p_1}{p_3^2} + 0 = \frac{p_1}{p_3^2} \\
s_{31} &= \frac{\partial x_3(p,w)}{\partial p_1} + \frac{\partial x_3(p,w)}{\partial w} x_1(p,w) = 0 + \frac{1}{p_3}\frac{p_2}{p_3} = \frac{p_2}{p_3^2} \\
s_{32} &= \frac{\partial x_3(p,w)}{\partial p_2} + \frac{\partial x_3(p,w)}{\partial w} x_2(p,w) = 0 + \frac{1}{p_3}\left(-\frac{p_1}{p_3}\right) = -\frac{p_1}{p_3^2} \\
s_{33} &= \frac{\partial x_3(p,w)}{\partial p_3} + \frac{\partial x_3(p,w)}{\partial w} x_3(p,w) = -\frac{w}{p_3^2} + \frac{1}{p_3}\frac{w}{p_3} = 0
\end{aligned}$$

Therefore, the Slutsky matrix is

$$S(p,w) = \begin{bmatrix} 0 & \frac{1}{p_3} & -\frac{p_2}{p_3^2} \\ -\frac{1}{p_3} & 0 & \frac{p_1}{p_3^2} \\ \frac{p_2}{p_3^2} & -\frac{p_1}{p_3^2} & 0 \end{bmatrix}$$