Recitation #2 (August 30th, 2019)

1. **[Checking properties of the Cobb-Douglas utility function.]** Consider the utility function

$$u(x) = \prod_{i=1}^{n} x_i^{\alpha_i},$$

where x denotes a vector of n different goods $x \in \mathbb{R}_+^n$, and $1 > \alpha_i > 0$. Check if this utility function satisfies: additivity, homogeneity of degree k, and homotheticity.

- First, note that this utility function is just a generalization of the Cobb-Douglas utility function to n goods. Indeed, for $n = 2$ goods

$$u(x) = \prod_{i=1}^{n} x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2}, \text{ where } \alpha_1, \alpha_2 > 0$$

- *Additivity.* The utility function is not additive, since the marginal utility of additional amounts of good k, $u_k(x)$, is

$$u_k(x) = \frac{\alpha_k}{x_k} \left(\prod_{i=1}^{n} x_i^{\alpha_i} \right)$$

and therefore, depends on the amounts of other goods consumed, i.e., $\frac{\partial u_k(x)}{\partial x_j} \neq 0$. [For a utility function to be additive, we should have obtained that the marginal utility of good k, $u_k(x)$ is independent on the amount of other goods, i.e., $\frac{\partial u_k(x)}{\partial x_j} = 0$ for all $j \neq k$.]

  - This can also be confirmed for the case of $n = 2$ goods, $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$, where the marginal utility of good 1 is

$$u_1(x) = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} = \frac{\alpha_1}{x_1} x_1^{\alpha_1} x_2^{\alpha_2}$$

  which depends on both x_1 and x_2. For this utility function to be additive, we should have obtained that $u_1(x)$ does not depend on x_2. A similar argument applies to $u_2(x)$.

- *Homogeneity.* Let us now check its degree of homogeneity. Simultaneously increasing all of its arguments by a common factor θ, we obtain

$$u(\theta x) = \prod_{i=1}^{n} (\theta x_i)^{\alpha_i} = \theta^{\sum_{i=1}^{n} \alpha_i} \left(\prod_{i=1}^{n} x_i^{\alpha_i} \right) = \theta^{\sum_{i=1}^{n} \alpha_i} u(x)$$

Therefore, utility function $u(x)$ is homogeneous of degree $\sum_{i=1}^{n} \alpha_i$. In the case of a Cobb-Douglas utility function for only two goods, $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$, we have that the degree of homogeneity is $\sum_{i=1}^{n=2} \alpha_i = \alpha_1 + \alpha_2$. Intuitively, when $\alpha_1 + \alpha_2 > 1 (< 1)$, a common increase in the amounts of all goods produces a more-than-proportional (less-than-proportional, respectively) increase in the consumer's utility level. If $\alpha_1 + \alpha_2 = 1$, then a common increase, θ, in the consumption of all goods, generates a proportional increase in the utility level of this individual, i.e., his utility increases by exactly θ. A similar argument applies to n goods, where either $\sum_{i=1}^{n} \alpha_i > 1$ or < 1.

- *Homotheticity.* Finally, the utility function satisfies homotheticity since it is homogeneous, and thus we can apply a monotonic transformation $g(\cdot)$ on $u(x)$ that produces a homothetic function $v(x) \equiv g(u(x))$. Nonetheless, let us check for homotheticity as a practice. In particular, let us first find the marginal rate of substitution between any two goods $k \neq l$

$$MRS_{l,k} \equiv -\frac{\alpha_k}{\alpha_l}\frac{x_l}{x_k}$$

  As a consequence, the $MRS_{l,k}$ between two goods $k \neq l$ only depends on the ratio of these two goods that the consumer enjoys, but is independent on any other good h. Therefore $\frac{\partial MRS_{l,k}}{\partial x_h} \neq 0$, for any third good $h \neq k \neq l$, implying that the $MRS_{l,k}$ (the slope of the consumer's indifference curves) only depends on the proportion of good l and k that the individual consumes. Graphically, this implies that the slope of the indifference curve coincides along any ray from the origin (since rays from the origin maintain the ratio between x_l and x_k constant).

2. **[Finding Walrasian demands-I.]** Determine the Walrasian demand $x(p,w) = (x_1(p,w), x_2(p,w))$ and the indirect utility function $v(p,w)$ for each of the following utility functions in $\mathbb{R}^2_+$. Briefly describe the indifference curves of each utility function and find the marginal rate of substitution, $MRS_{1,2}(x)$. Please consider the following two points in your analysis.

   - *Existence:* First, in all three cases the budget set is compact (it is closed, since the bundles in the frontier are available for the consumer, and bounded). Additionally, all utility functions are continuous. Therefore, we can apply Weierstrass theorem to conclude that each of the utility maximization problems (UMPs) we consider has at least one solution.

   - *Binding constraints:* We know that, if preferences are locally non-satiated, then the budget constraint will be binding, i.e., the consumer will be exhausting all his wealth. We can easily check that these utility functions are increasing in both

x_1 and x_2, which implies monotonicity and, in turn, entails local non-satiation. Hence, we can assume thereafter that the budget constraint is binding.

(a) Cobb-Douglas utility function, $u(x) = x_1^3 x_2^4$.

1. This utility function is a Cobb-Douglas utility function, with smooth indifference curves that are bowed-in towards the origin. Regarding the marginal rate of substitution between goods x_1 and x_2, MRS_{x_1,x_2}, we have

$$MRS_{x_1,x_2} = \frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}} = \frac{3x_1^2 x_2^4}{4x_1^3 x_2^3} = \frac{3x_2}{4x_1}$$

2. The UMP is given by

$$\max_{x_1,x_2} u(x) = x_1^3 x_2^4$$

subject to

$$p_1 x_1 + p_2 x_2 \leq w$$

$$x_1, x_2 \geq 0$$

As mentioned above, the budget constraint will be binding. Furthermore, since the utility from consuming zero amounts of either of the goods is zero, i.e., $u(0, \cdot) = u(\cdot, 0) = 0$, and the consumer's wealth is strictly positive, $w > 0$, then it can never be optimal to consume zero amounts of either of the goods. Therefore, we do not need to worry about the nonnegativity constraints $x_1, x_2 \geq 0$, i.e., there are no corner solutions. The Lagrangian of this UMP is then

$$\mathcal{L}(x_1, x_2; \lambda) = x_1^3 x_2^4 - \lambda \left[p_1 x_1 + p_2 x_2 - w\right]$$

The first order conditions are:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= 3x_1^2 x_2^4 - \lambda p_1 = 0\\ \frac{\partial \mathcal{L}}{\partial x_2} &= 4x_1^3 x_2^3 - \lambda p_2 = 0\end{aligned}$$

Solving for λ on both first order conditions, we obtain

$$\frac{3x_1^2 x_2^4}{p_1} = \frac{4x_1^3 x_2^3}{p_2} \iff \frac{3x_2}{p_1} = \frac{4x_1}{p_2}$$

This is the well-known "equal bang for the buck" condition across goods at utility maximizing bundles. (Intuitively, the consumer adjusts his consumption of goods 1 and 2 until the point in which the marginal utility per dollar on good 1 coincides with that of good 2.) Using now the budget constraint (which is binding), we have

$$p_1x_1 + p_2x_2 = w \iff x_1 = \frac{w}{p_1} - \frac{p_2x_2}{p_1}$$

and substituting this expression of x_1 into the above equality, yields the Walrasian demand for good 2

$$\frac{3x_2}{p_1} = \frac{4}{p_2}\left(\frac{w}{p_1} - \frac{p_2x_2}{p_1}\right) \iff x_2 = \frac{4}{7}\frac{w}{p_2}$$

and similarly solving for x_1, we obtain the Walrasian demand for good 1,

$$\frac{3}{p_1}\left(\frac{w}{p_2} - \frac{p_1x_1}{p_2}\right) = \frac{4x_1}{p_2} \iff x_1 = \frac{3}{7}\frac{w}{p_1}$$

Hence, the Walrasian demand function is

$$x(p,w) = \left(\frac{3}{7}\frac{w}{p_1}, \frac{4}{7}\frac{w}{p_2}\right)$$

And the indirect utility function $v(p,w)$ is given by plugging the Walrasian demand of each good into the consumer's utility function, which provides us with his utility level in equilibrium, as follows:

$$v(p,w) = \left(\frac{3}{7}\frac{w}{p_1}\right)^3 \left(\frac{4}{7}\frac{w}{p_2}\right)^4 = \frac{3^3 \times 4^4}{7^7}\frac{w^7}{p_1^3p_2^4}$$

(b) Preferences for substitutes (linear utility function), $u(x) = 3x_1 + 4x_2$.

- In order to draw indifference curves for this utility function, just consider some fixed utility level, e.g., $\overline{u} = 10$, and then solve for x_2, obtaining $x_2 = \frac{\overline{u}}{4} - \frac{3}{4}x_1$. Note that the resulting expressions are functions of x_1 only, and importantly, they are *linear* in x_1; as depicted in figure 1. Intuitively, this indicates that both goods can be substituted at the same rate, regardless of the amount of goods the consumer owns of every good (goods are perfect substitutes). The MRS_{x_1,x_2} confirms this intuition, since it is constant for any amount of x_1 and x_2,

$$MRS_{x_1,x_2} = \frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}} = \frac{3}{4}$$

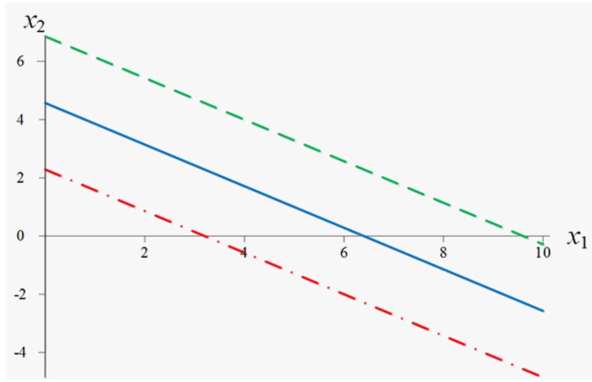

Figure 1. Indifference curves of $u(x) = 3x_1 + 4x_2$.

The $MRS_{x_1,x_2} = \frac{p_1}{p_2}$ condition in this case entails $\frac{3}{4} = \frac{p_1}{p_2}$, or $\frac{3}{p_1} = \frac{4}{p_2}$, which represents in the left (right) the marginal utility per dollar spent on good 1 (good 2, respectively), i.e., the "bang for the buck" on each good. When $\frac{3}{p_1} > \frac{4}{p_2}$, the consumer seeks to purchase good 1 alone, giving rise to a corner solution with $x_1(p,w) = \frac{w}{p_1}$ and $x_2(p,w) = 0$ as Walrasian demands. Similarly, when $\frac{3}{p_1} < \frac{4}{p_2}$ a corner solution emerges with only good 2 being consumed, i.e., $x_1(p,w) = 0$ and $x_2(p,w) = \frac{w}{p_2}$. Finally, when $\frac{3}{p_1} = \frac{4}{p_2}$, a continuum of equilibria arise as the consumer is indifferent between dedicating more money into good 1 or good 2; that is, all (x_1, x_2)-pairs on the budget line $p_1x_1 + p_2x_2 = w$ are utility-maximizing bundles.

- For completeness, we next show that we can obtain the same solutions if we were to set up the consumer's UMP, his associated Lagrangian, and take Kuhn-Tucker conditions. The UMP in this case is

$$\max_{x_1,x_2} u(x) = 3x_1 + 4x_2$$

subject to

$$p_1x_1 + p_2x_2 \leq w$$

$$x_1, x_2 \geq 0$$

As mentioned above, the budget constraint will be binding. The nonnegativity constraints, however, will not necessarily be binding, implying that in certain cases the consumer might choose to select zero amounts of some good. Therefore, we face a maximization problem with inequality constraints, $x_1, x_2 \geqslant 0$, and hence must use Kuhn-Tucker conditions. First, we set up the

Kuhn-Tucker style Lagrangian of this UMP

$$\mathcal{L}(x_1, x_2; \lambda_1, \lambda_2, \lambda_3) = 3x_1 + 4x_2 - \lambda_1 \left[p_1 x_1 + p_2 x_2 - w\right] + \lambda_2 x_1 + \lambda_3 x_2$$

The Kuhn-Tucker conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= 3 - \lambda_1 p_1 + \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= 4 - \lambda_1 p_2 + \lambda_3 = 0 \\
p_1 x_1 + p_2 x_2 &\leq w \\
x_1, x_2 &\geq 0 \\
\lambda_1 \left[p_1 x_1 + p_2 x_2 - w\right] &= 0, \\
\lambda_2 x_1 &= 0, \text{ and } \lambda_1 x_2 = 0
\end{aligned}$$

While we include all Kuhn-Tucker conditions for this type of maximization problems, some of them can be eliminated, since we know that the budget constraint is binding, i.e., $p_1 x_1 + p_2 x_2 = w$. Additionally, solving for λ_1 in the first two expressions, we obtain

$$\frac{3 + \lambda_2}{p_1} = \frac{4 + \lambda_3}{p_2} \qquad ((1))$$

Now we are ready to consider the solutions that can arise in the four possible cases in which the nonnegativity constraints can be met. These cases are

1. $\lambda_2 = 0$ and $\lambda_3 = 0$ i.e., $x_1 > 0$ and $x_2 > 0$
2. $\lambda_2 = 0$ and $\lambda_3 \neq 0$ i.e., $x_1 > 0$ and $x_2 = 0$
3. $\lambda_2 \neq 0$ and $\lambda_3 = 0$ i.e., $x_1 = 0$ and $x_2 > 0$
4. $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$ i.e., $x_1 = 0$ and $x_2 = 0$

- *CASE 1.* Interior solution, $x_1 > 0$ and $x_2 > 0$, i.e., $\lambda_2 = 0$ and $\lambda_3 = 0$. This implies that equation (1) becomes

$$\frac{3}{p_1} = \frac{4}{p_2} \Longleftrightarrow \frac{p_1}{p_2} = \frac{3}{4}$$

Hence, we can only have an interior solution when the price ratio is exactly $\frac{3}{4}$. In such a case, the budget line totally overlaps a indifference curve with the same slope, $\frac{3}{4}$ (as depicted in figure 2), and the consumer can choose any consumption bundle on the budget line. In particular, any bundle (x_1, x_2)

satisfying $p_1x_1 + p_2x_2 = w$ is optimal as long as the price ratio is exactly $\frac{p_1}{p_2} = \frac{3}{4}$.

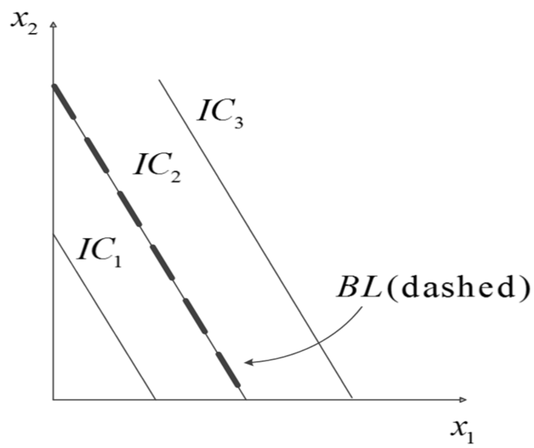

Figure 2. Case 1: Interior solutions

- *CASE 2.* Lower corner solution, $x_1 > 0$ and $x_2 = 0$, i.e., $\lambda_2 = 0$ and $\lambda_3 > 0$. Since wealth is fully spent on good 1, we have that $x_1 = \frac{w}{p_1}$. In order to determine when this solution is optimal, we can use expression (1), and the fact that $\lambda_2 = 0$, to obtain

$$\frac{3}{p_1} = \frac{4 + \lambda_3}{p_2} \iff \lambda_3 = 3\frac{p_2}{p_1} - 4$$

and since $\lambda_3 > 0$ we find that this solution is optimal when

$$\lambda_3 = 3\frac{p_2}{p_1} - 4 > 0 \iff \frac{p_1}{p_2} < \frac{3}{4}$$

Graphically, this happens when the linear indifference curves are steeper than the budget line, i.e., $MRS > \frac{p_1}{p_2}$, as depicted in figure 3.

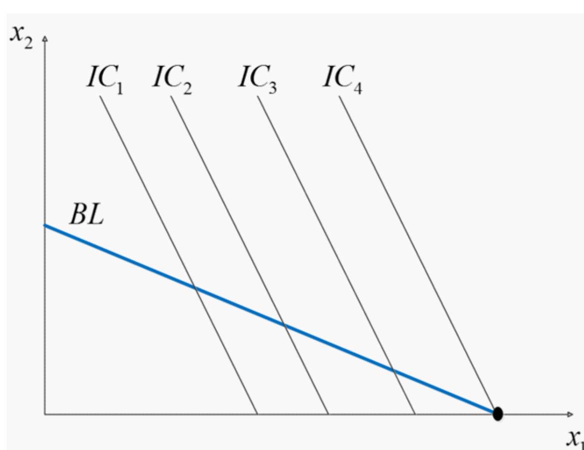

Figure 3. Case 2: Corner solution-I

- *CASE 3.* Upper corner solution, $x_1 = 0$ and $x_2 > 0$, i.e., $\lambda_2 > 0$ and $\lambda_3 = 0$. Since wealth is fully spent on good 2, we have that $x_2 = \frac{w}{p_2}$. In order to determine when this solution is optimal, we can use expression (1), and the fact that $\lambda_3 = 0$, which yields

$$\frac{3+\lambda_2}{p_1} = \frac{4}{p_2} \Longleftrightarrow \lambda_2 = 4\frac{p_1}{p_2} - 3$$

and since $\lambda_2 > 0$ we find that this solution becomes optimal when

$$\lambda_2 = 4\frac{p_1}{p_2} - 3 > 0 \Longleftrightarrow \frac{p_1}{p_2} > \frac{3}{4}$$

Graphically, this occurs when the linear indifference curves are flatter than the budget line, i.e., $MRS < \frac{p_1}{p_2}$, as illustrated figure 4.

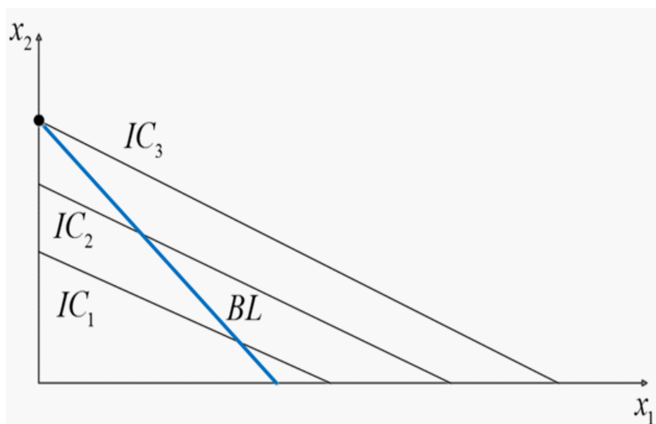

Figure 4. Case 3: Corner solution-II

- *CASE 4.* Corner solution, $x_1 = 0$ and $x_2 = 0$, i.e., $\lambda_2 > 0$ and $\lambda_3 > 0$. This solution would not exhaust this individual's wealth, i.e., it would imply $p_1x_1 + p_2x_2 < w$. Indeed, since the utility function is monotone, the budget constraint should be binding (the consumer should spend his entire wealth). Hence, case 4 cannot arise under any positive price-wealth pairs (p, w).
- *SUMMARY.* We can now summarize the Walrasian demand correspondence

$$x(p,w) = \begin{cases} \left(0, \frac{w}{p_2}\right) \text{ if } \frac{p_1}{p_2} > \frac{3}{4} \\ \text{any } (x_1, x_2) \in \mathbb{R}^2_+ \text{ s.t. } p_1x_1 + p_2x_2 = w \text{ if } \frac{p_1}{p_2} = \frac{3}{4}, \\ \left(\frac{w}{p_1}, 0\right) \text{ if } \frac{p_1}{p_2} < \frac{3}{4} \end{cases}$$

- Plugging this Walrasian demand into the utility function, yields the indirect utility function

$$v(p, w) = \begin{cases} 4\frac{w}{p_2} \text{ if } \frac{p_1}{p_2} \geq \frac{3}{4} \\ 3\frac{w}{p_1} \text{ if } \frac{p_1}{p_2} < \frac{3}{4} \end{cases}$$