Homework #3 (Due on September 11th, 2019)

1. Show that if the demand function $x_i(p, w)$ satisfies the uncompensated law of demand (ULD), then the matrix of derivatives $D_p x_i(p, w)$ is negative semidefinite. [*Hint:* Totally differentiate the Walrasian demand function $x_i(p, w)$ with respect to p and w, and use the uncompensated law of demand to cancel some terms. Justify.]

If $x_i(p, w)$ satisfies the uncompensated law of demand (ULD), then we have that for any change in prices dp, the matrix $D_p x_i(p, w)$ is negative semidefinite. That is we need to show that

$$dp \cdot D_p x_i(p, w) dp \leq 0$$

Proof: First, totally differentiating the Walrasian demand function $x_i(p, w)$ we obtain

$$dx_i(p, w) = D_p x_i(p, w) dp + D_w x_i(p, w) dw$$

But given that we are assuming the ULD property, we know that any change in prices is not going to be compensated by a change in the wealth level of the individual (in order to maintain him on the original indifference curve). Hence, $dw = 0$, and the above expression reduces to

$$dx_i(p, w) = D_p x_i(p, w) dp$$

Pre-multiplying both sides of this expression by dp, we have

$$dp \cdot dx_i(p, w) = dp \cdot D_p x_i(p, w) dp$$

Finally, by the ULD property we know that $dp \cdot dx_i(p, w) \leq 0$, which helps us conclude that $dp \cdot D_p x_i(p, w) dp \leq 0$. Therefore, matrix $D_p x_i(p, w)$ is negative semidefinite.

2. Suppose that demand and utility functions are differentiable, and preferences are homothetic. Show that:

   (a) $D_p x_i(p, w_i) = S_i(p, w_i) - \frac{1}{w_i} x_i(p, w_i) \cdot x_i(p, w_i)^T$

   - From the Slutsky equation we know that

   $$S_i(p, w) = D_p x_i(p, w_i) + D_w x_i(p, w_i) x_i(p, w_i)^T$$

   Furthermore, given homothetic preferences, we know that the Walrasian demand is defined by $x_i(p, w_i) = \alpha_i w_i$. Hence, solving for α_i we obtain

$\alpha_i = \frac{x_i(p,w_i)}{w}$. Additionally, the effect of an increase in the individual's wealth is given by $D_w x_i(p, w_i) = \alpha_i$. Combining both results we have that

$$D_w x_i(p, w_i) = \frac{x_i(p, w_i)}{w}$$

Plugging this result back into the Slutsky equation,

$$S_i(p, w) = D_p x_i(p, w_i) + \underbrace{\frac{x_i(p, w_i)}{w}}_{D_w x_i(p,w_i)} x_i(p, w_i)^T$$

and rearranging terms

$$D_p x_i(p, w_i) = S_i(p, w) - \frac{x_i(p, w_i)}{w} x_i(p, w_i)^T$$

which is exactly what we needed to show.

(b) $dp \cdot S_i(p, w_i) \cdot dp = 0$ when $dp = \alpha p$ (when the price change dp is proportional to the initial price level).

- From part (a) we know that

$$S_i(p, w) = D_p x_i(p, w_i) + \frac{x_i(p, w_i)}{w} x_i(p, w_i)^T$$

Pre- and post-multiplying every term by $dp = \alpha p$, we obtain

$$\alpha p \cdot S_i(p, w) \cdot \alpha p = \alpha p \cdot D_p x_i(p, w_i) \cdot \alpha p + \alpha p \cdot \frac{x_i(p, w_i)}{w} x_i(p, w_i)^T \cdot \alpha p$$

and canceling out the α's on both sides, we obtain the following expression, where we will separately focus on terms 1 and 2:

$$p \cdot S_i(p, w) \cdot p = \underbrace{p \cdot D_p x_i(p, w_i)}_{\text{Term 1}} \cdot p + \underbrace{p \cdot \frac{x_i(p, w_i)}{w}}_{\text{Term 2}} x_i(p, w_i)^T \cdot p$$

Term 1: From Cournot aggregation, we know that

$$p \cdot D_p x_i(p, w_i) + x_i(p, w_i)^T = 0 \Longleftrightarrow p \cdot D_p x_i(p, w_i) = -x_i(p, w_i)^T$$

Term 2: From Engel aggregation, we have that

$$p \cdot D_w x_i(p, w_i) = 1$$

which in our case is $p \cdot \frac{x_i(p,w_i)}{w} = 1$ since preferences are homothetic. Substituting the values of Terms 1 and 2 into the above Slutsky equation yields

$$p \cdot S_i(p,w) \cdot p = \underbrace{-x_i(p,w_i)^T}_{\text{Term 1}} \cdot p + \underbrace{1}_{\text{Term 2}} \cdot x_i(p,w_i)^T \cdot p = 0$$

Hence, $p \cdot S_i(p,w) \cdot p = 0$. We can then conclude that when price changes are proportional to the initial price level, $dp = \alpha p$, then $dp \cdot S_i(p,w_i) \cdot dp = 0$.

(c) $dp \cdot x_i(p,w_i) > 0$ when $dp = \alpha p$ (when the price change dp is proportional to the initial price level).

- By Walras' law, we know that

$$p \cdot x_i(p,w_i) = w$$

If dp is proportional to the price level, $dp = \alpha p$, then solving for p yields $p = \frac{dp}{\alpha}$. Substituting for this value of p in the above expression

$$\frac{dp}{\alpha} \cdot x_i(p,w_i) = w \text{ for any } \alpha > 0$$

and rearranging yields

$$dp \cdot x_i(p,w_i) = \alpha w$$

Since both $\alpha > 0$ and $w > 0$, then $\alpha w > 0$. Therefore, since that the right-hand side of the equality is positive, the left-hand side must also be positive, i.e., $dp \cdot x_i(p,w_i) > 0$.

3. Consider an individual facing price vector $p = (p_1, p_2) >> 0$ and income $w > 0$. If, after solving his UMP, his indirect utility function is $v(p,w) = \left(p_1^{\alpha} p_2^{1-\alpha}\right) w$, show that his utility function $u(x)$ must have a Cobb-Douglas representation, where $x = (x_1, x_2)$.

- *Proof (EMP approach).* First, recall from duality that

$$u(x) \equiv \min_{p \in \mathbb{R}_+^2} v(p, p \cdot x) \tag{1}$$

Since $v(p,w)$ is homogeneous of degree zero, we can divide both arguments by $p \cdot x$ to obtain that the indirect utility function $v(p,w)$ is unaffected, i.e., $v(p,w) = v\left(\frac{p}{p \cdot x}, 1\right)$. Let $\overline{p} \equiv \frac{p}{p \cdot x}$, and thus $v(p,w) = v(\overline{p}, 1)$. As a consequence, if price vector p^* minimizes $v(p, p \cdot x)$ for an income level $p \cdot x = w$, then price vector $\overline{p}$ minimizes $v(p,1)$ for an income level $p \cdot x = 1$. That is, we can rewrite program

(1) as

$$u(x) \equiv \min_{p \in \mathbb{R}^2_+} v(p,1) \quad \text{subject to } p \cdot x = 1 \tag{2}$$

- We can now find the price vector $p = (p_1, p_2)$ that solves program (2). Plugging them afterwards in the indirect utility function $v(p,1)$ will yield the original utility function $u(x)$ that this consumer maximized in his UMP (as stated in (2)). Since program (2) is a constrained minimization problem, we set up the Lagrangian

$$\mathcal{L} = \left(p_1^{\alpha} p_2^{1-\alpha}\right) - \lambda(p_1 x_1 + p_2 x_2 - 1)$$

Taking first-order conditions with respect to p_1 and p_2 yields, respectively

$$\begin{aligned} \frac{d\mathcal{L}}{dp_1} &= \alpha p_1^{\alpha-1} p_2^{1-\alpha} - \lambda x_1 = 0 \\ \frac{d\mathcal{L}}{dp_2} &= (1-\alpha) p_1^{\alpha} p_2^{-\alpha} - \lambda x_2 = 0 \end{aligned}$$

and

$$\frac{d\mathcal{L}}{d\lambda} = -p_1 x_1 - p_2 x_2 + 1 = 0$$

and simultaneously solving for p_1 and p_2 we obtain

$$p_1^* = \frac{\alpha}{x_1} \quad \text{and} \quad p_2^* = \frac{1-\alpha}{x_2}$$

We can finally plug these two prices, which solve (2), into the indirect utility funciton $v(p,1)$, yielding

$$v(p_1^*, p_2^*, 1) = \left(\left(\frac{\alpha}{x_1}\right)^{\alpha} \left(\frac{1-\alpha}{x_2}\right)^{1-\alpha} \right)(1) = \underbrace{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}_{\text{constant}} x_1^{-\alpha} x_2^{\alpha-1}$$

4. Consider a consumer with the following expenditure function

$$e(p, u^0) = g(p) + \left[u^0 \times f(p)\right]$$

where functions $g(p)$ and $f(p)$ depend on the price vector p alone. Show that a 1% increase in wealth leads to exactly a 1% increase in consumption (i.e., the income elasticity, $\varepsilon_{x_i,w}$) converges to one when the consumer's wealth level tends to infinity, i.e., $\lim_{w \to \infty} \varepsilon_{x_i,w} = 1$.

- The income-elasticity of good i is given by

$$\varepsilon_{x_i,w} = \frac{\partial x_i}{\partial w}\frac{w}{x_i}.$$

We hence need to first find the Walrasian demand associated to this expenditure function. In order to find it, we need to do it in two steps: first, use the expenditure function $e(p, u^0)$ to obtain the indirect utility function $v(p, w)$; and second, use the indirect utility function $v(p, w)$ to obtain the Walrasian demand of good i, $x_i(p, w)$.

- **First step.** *From $e(p, u^0)$ to $v(p, w)$.* Recall that, by the duality theorem, the minimal expenditure needed in order to reach the utility level that the consumer attains after solving his UMP, $v(p, w)$, is $e(p, v(p, w)) = w$, which in this exercise entails

$$g(p) + [v(p, w) \times f(p)] = w,$$

and solving for $v(p, w)$, we find that the indirect utility function is

$$v(p, w) = \frac{w - g(p)}{f(p)}.$$

- **Second step.** *From $v(p, w)$ to $x_i(p, w)$.* Once we found the consumer's indirect utility function, $v(p, w)$, we can now use Roy's identity to obtain the Walrasian demand,

$$\begin{aligned} x_i(p, w) &= -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\frac{-g_i(p)f(p) - f_i(p)[w-g(p)]}{f(p)^2}}{\frac{1}{f(p)}} = \\ &= g_i(p) + \frac{f_i(p)}{f(p)}[w - g(p)] \end{aligned}$$

where, for compactness, we denote $g_i(p) \equiv \frac{\partial g(p)}{\partial p_i}$ and similarly $f_i(p) \equiv \frac{\partial f(p)}{\partial p_i}$.

- After finding the Walrasian demand $x_i(p, w)$, we can identify the income elasticity of good i, as follows

$$\begin{aligned} \varepsilon_{x_i,w} &= \frac{\partial x_i}{\partial w}\frac{w}{x_i} = \frac{f_i(p)}{f(p)} \frac{w}{g_i(p) + \frac{f_i(p)}{f(p)}[w - g(p)]} = \\ &= \frac{f_i(p)w}{g_i(p)f(p) + f_i(p)w - f_i(p)g(p)} \end{aligned}$$

and taking the limit of this ratio when $w \to \infty$, we obtain that $\lim_{w\to\infty} \varepsilon_{x_i,w} = 1$.