Homework # 2 EconS501 [Due on September 4th, 2019]

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1. Provide an example where a choice structure, $C(B)$, satisfies WARP but preferences are not rational.

Suppose that $X = \{x, y, z\}$, the budget set is $B = \{\{x, y\}, \{y, z\}, \{x, z\}\}$ and the choice structure is the following

$$\begin{aligned} C(\{x, y\}) &= \{x\} \\ C(\{y, z\}) &= \{y\} \\ C(\{x, z\}) &= \{z\} \end{aligned}$$

that satisfies WARP by definition. However, we cannot have rationalizing prefences since the choice under $\{x, y\}$ and $\{y, z\}$ indicates that $x \succ y$ and $y \succ z$. Hence, by transitivity we should have that $x \succ z$, which contradicts the choice behavior under $C(\{x, z\}) = \{z\}$. Hence, preferences are not rational.

2. Show that a monotonic transformation of $u(x, y)$ does not necessarily satisfy the property of diminishing marginal utility.

Consider a positive monotonic transformation of $u(x, y)$ with the monotonic transformation function $F(\cdot)$:

$$V(x, y) = F(u(x, y)); \;\; F' > 0 \text{ and } F'' \lesseqqgtr 0$$

where x and y are two choices facing a consumer. The property of positive monotonic transformation is reflected by the first derivative being positive, that is, F', which means that all rankings of bundle (x, y) in the original utility are preserved by the monotonic transformation. however, $F'' \lesseqqgtr 0$, does not affect the positive monotonic transformation, so it can be positive, negative or zero.

The two utility functions then represent the same utility and, hence, the same preferences. Taking the first and second derivatives for the two functions we obtain

$$\begin{aligned} V_x &= F'u_x, \; V_y = F'u_y & (1) \\ V_{xx} &= F'u_{xx} + F''u_x u_x, \; V_{yy} = F'u_{yy} + F''u_y u_y & (2) \\ V_{xy} &= F'u_{xy} + F''u_x u_y, \;\; V_{yx} = F'u_{yx} + F''u_y u_x \end{aligned}$$

where u_x and u_y denotes the first derivative of the utility function $u(x, y)$ with respect to x and y, u_{ij} with $i, j = x, y$ denote the second derivatives (same notation for the monotonic transformation function). The first equation indicates that the sign of u_x and V_x (u_y and V_y) coincides. This means that the sign of the marginal utility of x and y remains the same after the monotonic transformation. However, the second equation indicates that it is possible that the sign of u_{xx} and V_{xx} (u_{yy} and V_{yy}) does not coincide. Note that $F' > 0$ and $F'' \lesseqgtr 0$, hence, if the original utility function satisfies the law of diminishing marginal utility ($u_{xx} < 0$). the transformed utility function does not necessarily satisfies this property since it depends on the sign and size of $F'' u_x u_x$. That is, the sign of u_{xx} can be different from that of V_{xx}.

3. Consider a consumer with CES utility function

$$u(x_1, x_2) = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}$$

where coefficient ρ satisfies $\rho \neq 0$ and $\rho \leq 1$.

(a) Find the Walrasian demands of this consumer, $x_1(p, w)$ and $x_2(p, w)$.

- The Lagragian in this individual's UMP is

$$\mathcal{L}(x_1, x_2; \lambda) = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}} + \lambda [w - p_1 x_1 - p_2 x_2]$$

Taking first order conditions yields

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= [x_1^\rho + x_2^\rho]^{\frac{1-\rho}{\rho}} x_1^{\rho - 1} - \lambda p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= [x_1^\rho + x_2^\rho]^{\frac{1-\rho}{\rho}} x_2^{\rho - 1} - \lambda p_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= w - p_1 x_1 - p_2 x_2 = 0
\end{aligned}$$

Rearranging the first two equalities, we obtain

$$x_2 = x_1 \left(\frac{p_2}{p_1} \right)^{\frac{1}{\rho - 1}}.$$

Plugging this result into the budget constraint, we obtain

$$p_1 x_1 - p_2 \left[x_1 \left(\frac{p_2}{p_1} \right)^{\frac{1}{\rho - 1}} \right] = w$$

and solving for x_1, we find the Walrasian demand for good 1,

$$x_1(p,w) = \frac{w \cdot p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}$$

For compactness, this demand can be expressed as $x_1(p,w) = \frac{w \cdot p_1^{r-1}}{p_1^r + p_2^r}$ where $r \equiv \frac{\rho}{\rho-1}$. We can finally plug $x_1(p,w)$ into $x_2 = x_1 \left(\frac{p_2}{p_1}\right)^r$ in order to find the Walrasian demand of good 2,

$$x_2(p,w) = \frac{w \cdot p_2^{r-1}}{p_1^r + p_2^r}$$

(b) What is the Walrasian demand of any good $i = \{1,2\}$ when parameter $\rho \to 0$?

- When $\rho \to 0$ (the consumer's preferences can be represeneted with a Cobb-Douglas utility function) parameter r also approaches zero. In this setting, his Walrasian demand for good i, $x_i(p,w) = \frac{w \cdot p_i^{r-1}}{p_i^r + p_j^r}$, becomes

$$\lim_{\rho \to 0} x_i(p,w) = \frac{w \cdot p_i^{-1}}{1+1} = \frac{w}{2p_i}$$

which exactly coincides with the Walrasian demand of a consumer with Cobb-Douglas utility function $u(x_1,x_2) = x_1^\alpha x_2^\alpha$ for any $\alpha > 0$.

4. Consider the following demand functions for a consumer facing K goods, and where p_i denotes the price of good i, and p_{-i} represents the vector of prices different from p_i, i.e., $p_{-i} = (p_1, p_2, ..., p_{i-1}, p_{i+1}, ..., p_K)$. State conditions under which these demand functions satisfy Walras' Law.

(a) $x_i\left(p_i, p_{-i}, w\right) = \frac{1}{p_i}\left[\alpha_i + \beta_i w + \sum_{j=1}^{K} \gamma_{ij} p_j\right]$

- First, recall that for Walras' Law to be satisfied, we need to show that wealth is entirely spent on the consumption of all K goods. That is, for a strictly positive vector of prices ($p >> 0$) and positive wealth level ($w > 0$), $p \cdot x = w$, or alternatively, $\sum_{i=1}^{K} p_i x_i = w$.
- For the Walrasian demand in this exercises, and summing over all K goods, we obtain

$$\sum_{i=1}^{K} x_i\left(p_i, p_{-i}, w\right) p_i = \sum_{i=1}^{K} \alpha_i + \sum_{i=1}^{K} \beta_i w + \sum_{i=1}^{K}\sum_{j=1}^{K} \gamma_{ij} p_j$$

Rearranging, we can see that Walras' Law is satisfied if

$$\sum_{i=1}^{K}\alpha_i + \sum_{i=1}^{K}\sum_{j=1}^{K}\gamma_{ij}p_j + w\sum_{i=1}^{K}\beta_i = w$$

which holds if parameters α_i and γ_{ij} satisfy $\sum_{i=1}^{K}\alpha_i = 0$ and $\sum_{j=1}^{K}\gamma_{ij} = 0$, and if β_i satisfies $\sum_{i=1}^{K}\beta_i = 1$. (For instance, if $\alpha_i = \gamma_{ij} = 0$ for all i and j, and if $\beta_i = \frac{1}{K}$ for all good i, Walras' Law holds.)

(b) $x_i(p_i, p_{-i}, w) = \frac{w}{p_i}\left[\alpha_i + \beta_i \log w + \sum_{i=1}^{K}\gamma_{ij}\log p_j\right]$

- Rearranging,

$$p_i x_i = w\left[\alpha_i + \beta_i \log w + \sum_{i=1}^{K}\sum_{j=1}^{K}\gamma_{ij}\log p_j\right]$$

Summing over all K goods, Wlalras' Law holds if

$$\sum_{i=1}^{K}p_i x_i = w\left[\sum_{i=1}^{K}\alpha_i + \sum_{i=1}^{K}\beta_i \log w + \sum_{i=1}^{K}\sum_{j=1}^{K}\gamma_{ij}\log p_j\right] = w$$

Hence, Walras' Law is satisfied if the term in brackets is exactly equal to 1, which holds if $\sum_{i=1}^{K}\alpha_i = 1$, $\sum_{i=1}^{K}\beta_i = 0$ and $\sum_{i=1}^{K}\gamma_{ij} = 0$.

(c) $x_i(p_i, p_{-i}, w) = \frac{1}{p_i}\left[\alpha_i + \beta_i w + \gamma_i w^2\right]$

- Rearranging, $px_i = \alpha_i + \beta_i w + \gamma_i w^2$ and summing over all K goods,

$$\sum_{i=1}^{K}p_i x_i = \sum_{i=1}^{K}\alpha_i + w\sum_{i=1}^{K}\beta_i + w^2\sum_{i=1}^{K}\gamma_i$$

Hence, Walras' Law holds if

$$\sum_{i=1}^{K}\alpha_i + w^2\sum_{i=1}^{K}\gamma_i + w\sum_{i=1}^{K}\beta_i = w$$

That is, Walras' Law holds if parameters α_i and γ_i satisfy $\sum_{i=1}^{K}\alpha_i = \sum_{i=1}^{K}\gamma_i = 0$ and if β_i satisfies $\sum_{i=1}^{K}\beta_i = 1$. (Similarly as in part (a) of this exercise, these conditions hold if, for instance, $\alpha_i = \gamma_i = 0$ for all good i, and if $\beta_i = \frac{1}{K}$ for all i. Walras' Law can nonetheless be satisfied under less extreme assumptions on these parameters. For instance, in the case of $K = 2$ goods, Walras' Law holds if $\alpha_i = -\alpha_j$, $\gamma_i = -\gamma_j$, and $\beta_i = 1 - \beta_j$ for all $j \neq i$.)

5. Check whether the following demand functions satisfy the Weak Axiom of Revealed Preference (WARP).

   (a) "Random demand": For any pair of prices p_1 and p_2 and wealth w, the consumer randomizes uniformly over all points in the budget frontier.[1]

   - Let us prove that this demand function does not necessarily satisfy WARP by using an example, as depicted in figure 1.

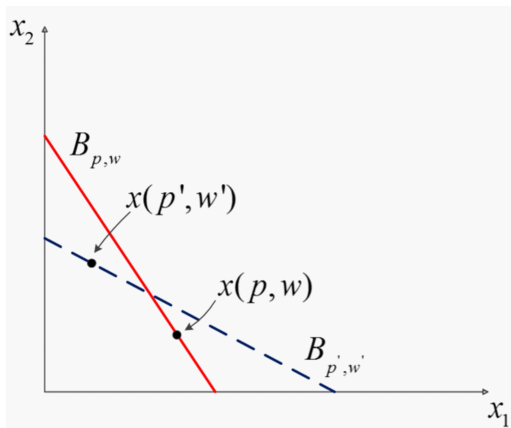

Figure 1. Random demand.

WARP states that

$$\text{if } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \text{ then } p' \cdot x(p, w) > w'$$

That is, if the new consumption bundle (chosen under the new prices and wealth) is affordable under the old prices and wealth, then it must be the case that the old consumption bundle is not affordable under the new prices and wealth. In the case of the random demand depicted in figure 2.4, however, we find that

$$p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \textbf{ but } p' \cdot x(p, w) < w'$$

That is, the old consumption bundle is *still* affordable under the new prices

[1]For instance, this demand can arise when the consumer regards two goods as perfect substitutes and their price ratio $\frac{p_1}{p_2}$ coincides with the ratio of marginal utilities. In this case, a continuum of Walrasian demands emerge, i.e., one for each point of the consumer's budget line. Since the consumer is indifferent among all these points, he can randomly choose one of these optimal bundles.

and wealth, i.e., graphically, bundle $x(p, w)$ lies below budget line $B_{p',w'}$ (dashed line in the figure). Since there exists a positive probability that random demand assigns bundles as the ones illustrated in figure 2.4, we can conclude that random demand does not satisfy WARP.

(b) "Average demand": The expected "random demand" given p_1, p_2 and w.[2]

- First, note that if the consumer randomizes uniformly over all points in her budget line (as described by the random demand), then the expected random demand is allocated at the midpoint of the budget line, as depicted in figure 2 for budget line $B_{p,w}$.

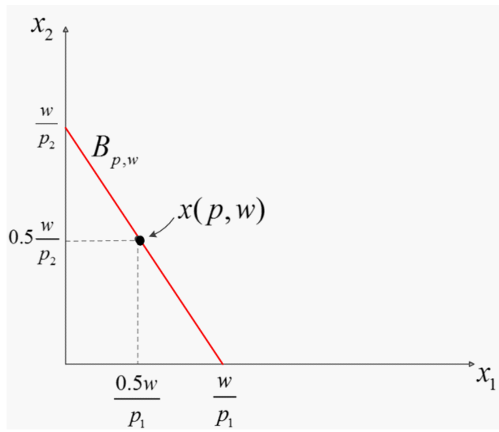

Figure 2. Average demand-I

Let us now prove that WARP is satisfied for average demand. Let us work by contradiction, by assuming that average demand violates WARP. There are two possibilities in which this violation might take place, as the two panels in figure 3 illustrate. In particular, in both figures bundle $x(p^{'}, w^{'})$ is affordable under old prices and wealth, i.e., it lies on or below budget line $B_{p,w}$, but bundle $x(p, w)$ is also affordable under the new prices and wealth, i.e., it lies

[2] If the "random demand" could arise in the presence of perfect substitutes, the "average demand" can then emerge as the average bundle that this consumer selects after randomly choosing a bundle from his budget line (as predicted by random demand) a sufficient number of times.

on or below $B_{p',w'}$, which constitute a violation of WARP.

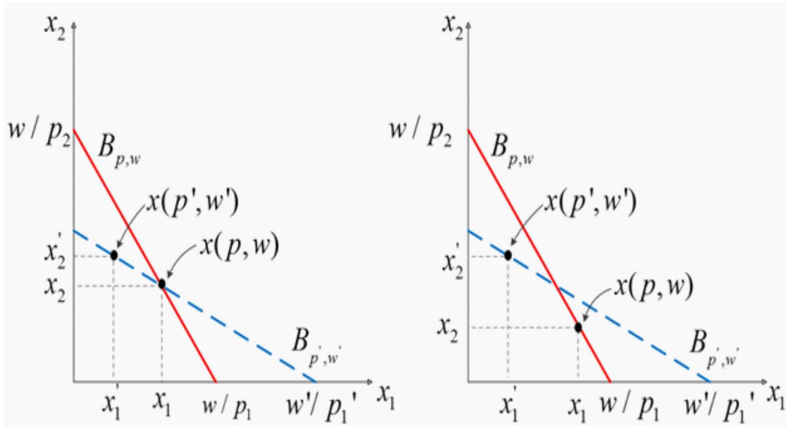

Figure 3. Average demand-II

- Let us first determine the location of points x_1 and x_1' according to the average demand. Recall that these points must to be located at the midpoint of the budget line. Hence,

$$x_1 = \frac{1}{2}\frac{w}{p_1} \text{ and } x_1' = \frac{1}{2}\frac{w'}{p_1'}$$

therefore $2x_1 = \frac{w}{p_1}$ and $2x_1' = \frac{w'}{p_1'}$. Moreover, we can see in both figures that $x_1' < x_1$. Therefore, $2x_1' < 2x_1$, which implies

$$\frac{w'}{p_1'} < \frac{w}{p_1}$$

But in both figures we actually see the opposite, i.e., $\frac{w'}{p_1'} > \frac{w}{p_1}$. Hence, we have reached a contradiction, and average demand cannot violate WARP.

(c) "Conspicuous demand": The individual spends all his wealth on the most expensive good. This is often referred to as "conspicuous consumption" and includes items such as luxury cars, yachts and private islands! For instance, if good 1 is the most expensive, $p_1 > p_2$, then the consumer spends all his wealth on good 1, $x_1(p_1, p_2, w) = \frac{w}{p_1}$, but nothing on good 2, $x_2(p_1, p_2, w) = 0$. More generally, for any p_1, p_2 and w, the demand for good $i = \{1, 2\}$ is

$$x_i\left(p_1, p_2, w\right) = \begin{cases} \frac{w}{p_i} \text{ if } \frac{w}{p_i} = \min\{\frac{w}{p_1}, \frac{w}{p_2}\} \text{ and } \frac{w}{p_1} \neq \frac{w}{p_2}, \\ \frac{w}{p_i} \text{ if } \frac{w}{p_1} = \frac{w}{p_2} \text{ and } i = 1 \\ 0 \text{ otherwise} \end{cases}$$

- Let us divide it into two cases: Case 1, in which $p_1 \geq p_2$ (good 1 is the most expensive), and Case 2, in which $p_1 < p_2$ (good 2 is the most expensive). Once we represent the conspicuous demand for all price vectors, we will check if it satisfies WARP.

- *Case 1* ($p_1 \geq p_2$): The demand function of good 1 is $x_1(p_1, p_2, w) = \frac{w}{p_1}$, while the demand of good 2 is zero, $x_2(p_1, p_2, w) = 0$, as depicted in figure 4.

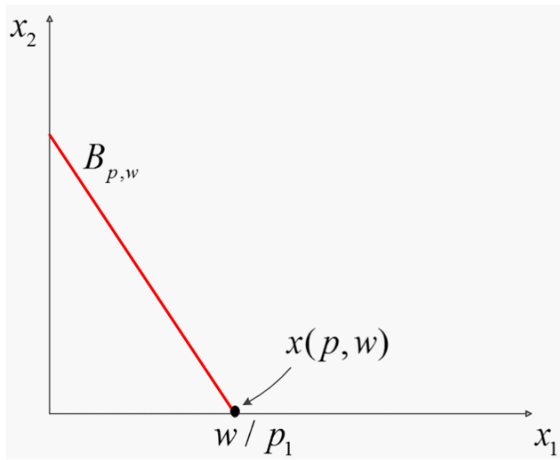

Figure 4. Conpicuous demand-I

- *Case 2* ($p_1 < p_2$): In this case, the demand function of good 2 reduces to $x_2(p_1, p_2, w) = \frac{w}{p_2}$,while that of good 1 is zero, as illustrated in figure 5.

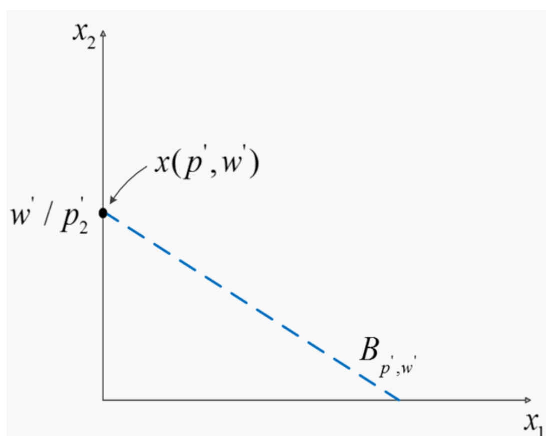

Figure 5. Conpicuous demand-II

- Summarizing, figure 2.9b represents both cases simultaneously, where note that budget line $B_{p,w}$ represents a higher price for p_1 (relative to p_2) while budget line $B_{p',w'}$ depicts the opposite case, where p_2 is high relative to p_1.

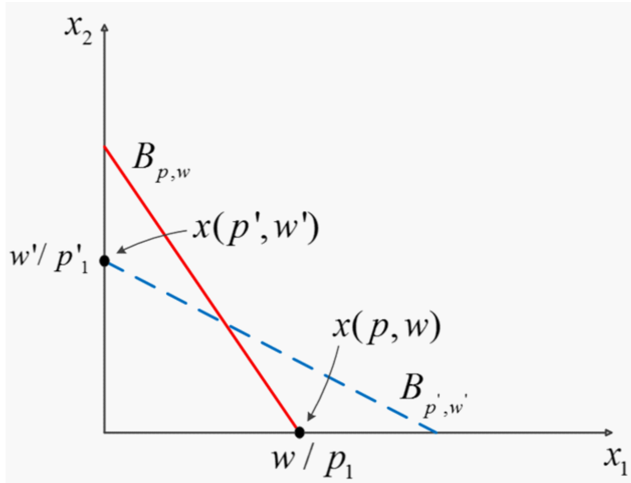

Figure 6. Conpicuous demand-III

- After depicting the bundles that represent a conspicuous demand for all price vectors, we can check if conspicuous demand satisfies WARP. As mentioned in part (a), WARP states that

$$\text{if } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \text{ then } p' \cdot x(p, w) > w'$$

Figure 2.9b summarizing both cases. While bundle $x(p', w')$ is affordable under old prices and wealth, i.e., $p \cdot x(p', w') \leq w$ since it lies below the solid budget line $B_{p,w}$ in figure 2.9b, bundle $x(p, w)$ is still affordable under new prices and wealth, i.e., $p' \cdot x(p, w) < w'$, since it lies below the dashed budget line $B_{p',w'}$. Intuitively, both bundles $x(p, w)$ and $x(p', w')$ are available at (p, w) and at (p', w'), but the consumer does not choose the same bundle under (p, w) than under (p', w'). Hence, conspicuous demand violates WARP.