Solution Homework 1 - EconS 501

1. **[Checking properties of preference relations-I].** In an alternative universe, dolls are anthropomorphized and chosen for a single child, while defective dolls are dropped into the remote town of Uglytown. Among these dolls is the idealistic Molly, who dreams being chosen for a child, despite Uglytown's Mayor Oy assuring her this is a myth. In order to be chosen by a kid they need to perform two tasks: (1) master the song skidamarink (task x) and (2) dress up like the perfect dollssing (task x). Molly is very motivated and she prefers a combination of these two tasks that contains more of dress up, i.e., $(x_1, y_1) \succsim (x_2, y_2)$ if and only if $y_1 \geq y_2 + 1$. For this preference relation in the performance of two tasks to become the perfect doll (tasks x and y): describe the upper contour set, the lower contour set, the indifference set of bundle $(2, 3)$, and interpret them. Then check whether this preference relation is rational (by separately examining whether they are complete and transitive), monotone, and convex.

   (a) Bundle (x_1, y_1) is weakly preferred to (x_2, y_2), i.e, $(x_1, y_1) \succeq (x_2, y_2)$ if and only if $y_1 \geq y_2 + 1$.

   - Let us first build some intuition on this preference relation. First, note that an individual prefers a bundle, (x_1, y_1) to another bundle, (x_2, y_2) if and only if the second component of the bundle y_1, contains at least one unit more than the second component of bundle y_2, i.e., $y_1 \geq y_2 + 1$. For instance, $(2, 3)$ is preferred to $(4, 2)$ since $y_1 = 3$ and $y_2 = 2$, implying $3 \geq 2 + 1 = 3$. Importantly, the individual ignores the content of the first component when comparing the two bundles. Let us next describe the upper contour, lower contour, and indifference set of a given bundle, such as $(2, 3)$. The upper contour set of this bundle is given by

     $$UCS\,(2,3) = \{(x_1, y_1) \succeq (2,3) \Longleftrightarrow y_1 \geq 3 + 1\} = \{(y_1, y_2) : y_1 \geq 4\}$$

     while the lower contour set is defined as

     $$LCS\,(2,3) = \{(2,3) \succeq (x_1, y_1) \Longleftrightarrow 3 \geq y_2 + 1\} = \{(y_1, y_2) : y_2 \leq 2\}$$

     Finally, the consumer is indifferent between bundle $(2, 3)$ and the set of bundles where

     $$IND\,(2,3) = \{(x_1, y_1) \sim (2,3) \Longleftrightarrow \varnothing\}$$

- *Completeness.* For this property to hold, we need that, for any pair of bundles (x_1, y_1) and (x_2, y_2), either $(x_1, y_1) \succeq (x_2, y_2)$ or $(x_2, y_2) \succeq (x_1, y_1)$, or both (i.e., $(x_1, y_1) \sim (x_2, y_2)$). Since for this preference relation the indifference set is empty then it is not complete.

- *Transitivity.* We need to show that, for any three bundles (x_1, y_1),(x_2, y_2) and (x_3, y_3) such that $(x_1, y_1) \succeq (x_2, y_2)$ and $(x_2, y_2) \succeq (x_3, y_3)$, then $(x_1, y_1) \succeq (x_3, y_3)$. This property holds for this preference relation. In order to show this result, notice that a bundle (x_1, y_1) is preferred to another bundle (x_2, y_2) if the second component of the first bundle y_1, is larger than that of the second bundle, y_2, by more than one unit, i.e., condition $y_1 \geq y_2 + 1$ is equivalent to $1 \leq y_1 - y_2$. A similar argument can be extended to the comparison between two bundles (x_2, y_2) and (x_3, y_3) where the former is preferred to the latter if and only if the distance between their second component is greater than one, i.e., $1 \leq y_2 - y_3$; as we next show with an example

  Consider the following three bundles,

$$(x_1, y_1) = (4, 6)$$

$$(x_2, y_2) = (1, 5)$$

$$(x_3, y_3) = (2, 4)$$

  First, note that $(x_1, y_1) \succeq (x_2, y_2)$ since the difference in their second component is greater (or equal) to one unit, $y_1 \geq y_2+1$(i.e., $6 \geq 5+1$). Additionally, $(x_2, y_2) \succeq (x_3, y_3)$ is also satisfied since $y_2 \geq y_3 + 1$ (i.e., $5 \geq 4+1$). Therefore $(x_1, y_1) \succeq (x_3, y_3)$ since the difference between y_1 and y_3 is larger than one unit, $y_1 \geq y_3 + 1$. Hence, this preference relation satisfies transitivity.

- *Monotonicity.* This property is satisfied for this preference relation. In particular, we need to show that increasing the amount of good 2 yields a new bundle $(x_1, y_1 + \varepsilon)$ that is weakly preferred to the original bundle (x_1, y_1), i.e., the comparison of their first component yields $y_1 + \varepsilon \geq y_1 + 1$, which holds iff $\varepsilon \geq 1$. Similarly, increasing the amount of the first component produces a new bundle $(x_1 + \varepsilon, y_1)$ which is weakly preferred to the original bundle (x_1, y_1). Recall that this individual compares bundles by evaluating the second component alone. Since in this case the amount of the first component is unaffected then he is indifferent between bundle $(x_1 + \varepsilon, y_1)$ and (x_1, y_1);

an indifference that is allowed by the definition of monotonicity. Hence, the preference relation does satisfies monotonicity.

- *Convexity.* This property implies that the upper contour set must be convex, that is, if bundle (x_1, y_1) is weakly preferred to (x_2, y_2), $(x_1, y_1) \succeq (x_2, y_2)$, then the convex combination of these two bundles is also weakly preferred to (x_2, y_2),

$$\lambda (x_1, y_1) + (1 - \lambda) (x_2, y_2) \succeq (x_2, y_2) \text{ for any } \lambda \epsilon \ (0, 1) .$$

In this case, $(x_1, y_1) \succeq (x_2, y_2)$ implies that $y_1 \geq y_2 + 1$ whereas $\lambda (x_1, y_1) + (1 - \lambda) (x_2, y_2) \succeq (x_2, y_2)$ implies

$$\lambda y_1 + (1 - \lambda) y_2 \geq y_2 + 1 \text{ for any } \lambda \epsilon \ (0, 1)$$

which simplifies to $\lambda (y_1 - y_2) \geq 1$. However, the premise from $(x_1, y_1) \succeq (x_2, y_2)$ i.e. $y_1 \geq y_2 + 1$ entails that $\lambda y_1 \geq \lambda y_2 + 1$ must also hold. (To see that, note that $y_2 \geq y_1 + 1$ can be written as $(y_1 - y_2) - 1 \geq 0$ while $\lambda y_1 \geq \lambda y_2 + 1$ can be expressed as $\lambda (y_1 - y_2) - 1 \geq 0$ where $\lambda (y_1 - y_2) - 1 \leq (y_1 - y_2) - 1$ since $\lambda \epsilon \ (0, 1)$. Therefore, $(y_1 - y_2) - 1 \geq 0$ is not a sufficient condition for $\lambda (y_1 - y_2) - 1 \geq 0$. We also need a more restrictive condition $(y_1 - y_2) \geq 2$. Hence, this preference relation is not convex since $(y_1 - y_2) \geq 1$. Example, consider $(2, 3)$ and $(2, 2)$ in this case $(x_1, y_1) \succeq (x_2, y_2)$, however, $0.5 \times 3 + (1 - 0.5) 2 \not\geq 2 + 1$.

2. **[Checking properties of preference relations-II].** Consider the following preference relation defined in $X = \mathbb{R}^2_+$. A bundle (x_1, x_2) is weakly preferred to another bundle (y_1, y_2), i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if

$$\min \{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min \{3y_1 + 2y_2, 2y_1 + 3y_2\}$$

(a) For any given bundle (y_1, y_2), draw the upper contour set, the lower contour set, and the indifference set of this preference relation. Interpret.

- Take a bundle $(2, 1)$. Then,

$$\min \{3 * 2 + 2 * 1, 2 * 2 + 3 * 1\} = \min \{8, 7\} = 7.$$

The upper contour set of this bundle is given by

$$\begin{aligned} UCS(2,1) &= \{(x_1, x_2) \succsim (2,1)\} \\ &= \{\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq 7 \equiv \min\{8,7\}\} \end{aligned}$$

which is graphically represented by all those bundles in $\mathbb{R}^2_+$ which are strictly above *both* lines $3x_1 + 2x_2 = 7$ and $2x_1 + 3x_2 = 7$. That is, for all (x_1, x_2) strictly above both lines

$$x_2 = \frac{7}{2} - \frac{3}{2}x_1 \text{ and } x_2 = \frac{7}{3} - \frac{2}{3}x_1.$$

(See figure 1.9, which depicts these two lines and shades the set of bundles lying weakly above both lines.)

- On the other hand, the lower contour set is defined as

$$\begin{aligned} LCS(2,1) &= \{(2,1) \succsim (x_1, x_2)\} \\ &= \{7 \geq \min\{3x_1 + 2x_2, 2x_1 + 3x_2\}\}, \end{aligned}$$

which is graphically represented by all bundles (x_1, x_2) weakly below the maximum of the lines described above. For instance, bundle $(y_1, y_2) = (2.5, 0)$, which lies on the horizontal axis and between both lines' horizontal intercept, implies

$$\min\{3 \cdot 2.5 + 2 \cdot 0, 2 \cdot 2.5 + 3 \cdot 0\} = \min\{7.5, 5\} = 5$$

thus implying that this consumer prefers bundle $(x_1, x_2) = (2,1)$ than $(y_1, y_2) = (2.5, 0)$. A similar argument applies to all other bundles lying above $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ and below $x_2 = \frac{7}{3} - \frac{2}{3}x_1$, where bundle $(2.5, 0)$ also belongs; see the triangle that both lines form at the right-hand side of the figure. Similarly, bundles such as $(0, 2.5)$ yield

$$\min\{3 \cdot 0 + 2 \cdot 2.5, 2 \cdot 0 + 3 \cdot 2.5\} = \min\{5, 7.5\} = 5,$$

which implies that the consumer also prefers bundle $(2,1)$ to $(0, 2.5)$. An analogous argument applies to all bundles above line $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ but below

$x_2 = \frac{7}{3} - \frac{2}{3}x_1$ in the triangle at the left-hand side of figure 2.1.

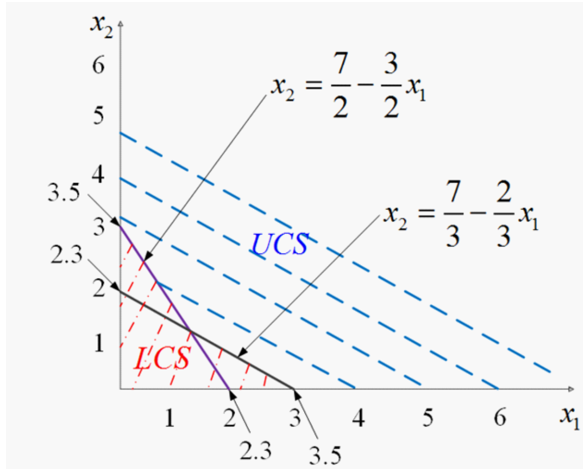

Figure 2.1. UCS and LCS of bundle (2,1).

Finally, those bundles for which the UCS and LCS overlap are those in IND of bundle (2,1).

(b) Check if this preference relation satisfies: (i) completeness, (ii) transitivity, and (iii) weak convexity.

- *Completeness.* First, note that both of the elements in the min$\{\cdot\}$ operator are real numbers, i.e., $(3x_1 + 2x_2) \in \mathbb{R}_+$ and $(2x_1 + 3x_2) \in \mathbb{R}_+$, thus implying that the minimum

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} = a$$

exists and it is also a real number, $a \in \mathbb{R}_+$. Similarly, the minimum

$$\min\{3y_1 + 2y_2, 2y_1 + 3y_2\} = b$$

exists and is a real number, $b \in \mathbb{R}_+$. Therefore, we can easily compare a and b, obtaining that either $a \geq b$, which implies $(x_1, x_2) \succsim (y_1, y_2)$; or $a \leq b$, which implies $(y_1, y_2) \succsim (x_1, x_2)$, or both, $a = b$, which entails $(x_1, x_2) \sim (y_1, y_2)$. Hence, the preference relation is complete.

- *Transitivity.* We need to show that, for any three bundles (x_1, x_2), (y_1, y_2) and (z_1, z_2) such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ implies

$$a \equiv \min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \equiv b$$

and $(y_1, y_2) \succsim (z_1, z_2)$ implies that

$$b \equiv \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \geq \min\{3z_1 + 2z_2, 2z_1 + 3z_2\} \equiv c$$

Combining both conditions we have that $a \geq b \geq c$, which implies that $a \geq c$. Hence, we have that

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3z_1 + 2z_2, 2z_1 + 3z_2\}$$

and thus $(x_1, x_2) \succsim (z_1, z_2)$, implying that this preference relation is transitive.

- *Weak Convexity.* This property implies that the upper contour set must be convex. That is, if bundle (x_1, x_2) is weakly preferred to (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$, then the convex combination of these two bundles is also weakly preferred to (y_1, y_2),

$$\lambda(x_1, x_2) + (1 - \lambda)(y_1, y_2) \succsim (y_1, y_2) \quad \text{for any } \lambda \in [0, 1]$$

For compactness, let $a \equiv 3x_1 + 2x_2$, $b \equiv 2x_1 + 3x_2$, $c \equiv 3y_1 + 2y_2$ and $d \equiv 2y_1 + 3y_2$. Hence, the property that $(x_1, x_2) \succsim (y_1, y_2)$ implies $\min\{a, b\} \geq \min\{c, d\}$. We therefore need to show that

$$\min\{\lambda a + (1 - \lambda) c, \lambda b + (1 - \lambda) d\} \geq \min\{c, d\}$$

1. *First case*: $\min\{a, b\} = a$, $\min\{c, d\} = c$ and $a \geq c$. Therefore,

$$\min\{\lambda a + (1 - \lambda) c, \lambda b + (1 - \lambda) d\} = \lambda a + (1 - \lambda) c$$

   and $\lambda a + (1 - \lambda) c > \min\{c, d\} = c$. For this case, convexity is satisfied.

2. *Second case*: $\min\{a, b\} = a$, $\min\{c, d\} = d$ and $a \geq d$. Hence, $a > b$ and $c > d$, implying that

$$\min\{\lambda a + (1 - \lambda) c, \lambda b + (1 - \lambda) d\} = \lambda a + (1 - \lambda) d$$

   and $\lambda a + (1 - \lambda) d \geq \min\{c, d\} = d$ given that $a \geq d$. For this case, convexity is satisfied as well. An analogous argument applies in the other two cases, in which $\min\{a,b\} = b$ and $\min\{c,d\} = c$, and in which $\min\{a,b\} = b$ but $\min\{c,d\} = d$.

3. [**Convexity and Strict Convexity.**] Explain convexity and strict convexity in preference relations, and compare them. Provide an example where a bundle x is preferred to bundle y when preferences satisfy convexity, but x is not necessarily preferred to y under strict convexity.

   (a) • **Convexity:** A preference relation satisfies convexity if, for any two bundle $x, y \in X, x \succsim y$ implies that $\lambda x + (1-\lambda) y \succsim y$ for all $\lambda \in (0,1)$. Intuitively, the combination of bundles x and y is weakly preferred to bundle y alone. Graphically, a preference relation satisfies convexity if, for every bundle x, its upper contour set, $UCS(x) = \{y \in X : y \in\} \succsim x\}$, is a convex set, and that indifference curves are bowed in toward the origin.

   • **Strict Convexity:** A preference relation satisfies strict convexity if, for every two bundles x and y weakly preferred to z, that is, for all $x, y \in X$, where $x \neq y$ such that $x \succsim z$ and $y \succsim z$, their convex combination $\lambda x + (1-\lambda) y$, where $\lambda \in (0,1)$, is strictly preferred to bundle z.Graphically, when bundle x and y lie on both the UCS and the IND, the definition of strict convexity is satisfied because the convex combination of these two bundles lies strictly inside the UCS, thus implying $\lambda x + (1-\lambda) y \succ z$.

   A preference relation that is represented by a utility function such as $u(x_1, x_2) = ax_1 + bx_2$, where $a, b > 0$,which indicates that goods 1 and 2 are regarded as substitutes illustrates a preference relation that satisfies convexity but not strict convexity. In that case, since the convex combination of bundle x and y lies strictly on the indifference curve, their convex combination therefore produces a bundle $\lambda x + (1-\lambda) y$ that the consumer regards as indifferent to bundle z. For another example, consider preferences over two goods that are regarded as perfect complements, represented by the utility function $u(x_1, x_2) = \min\{ax_1, bx_2\}$, where $a, b, > 0$ are parameters. The convex combination of bundle x and y produces a bundle that does not necessarily lie strictly within the UCS.

3. **Lexicographic preference relation.** Let us define a lexicographic preference relation in a consumption set $X \times Y$, as follows:

$$(x_1, x_2) \succsim (y_1, y_2) \text{ if and only if } \begin{cases} x_1 > y_1, \text{ or if} \\ x_1 = y_1 \text{ and } x_2 \geqslant y_2 \end{cases} \tag{1}$$

Intuitively, the consumer prefers bundle x to y if the former contains more units of the first good than the latter, i.e., $x_1 > y_1$. However, if both bundles contain the same amounts of good 1, $x_1 = y_1$, the consumer ranks bundle x above y if the former has more units of good 2 than the latter, i.e., $x_2 \geq y_2$. For simplicity, assume that both components have been normalized to $X = [0, 1]$ and $Y = [0, 1]$.

(a) Show that the lexicographic preference relation satisfies rationality (i.e., it is complete and transitive).

1. *Completeness.* By definition, $\succsim$ is a complete preference relation if for all bundles (x_1, x_2),$(y_1, y_2) \in \mathbb{R}^2$, either $(x_1, x_2) \succsim (y_1, y_2)$, or $(y_1, y_2) \succsim (x_1, x_2)$, or both. Hence, we need to show that

$$(x_1, x_2) \not\succsim (y_1, y_2) \implies (y_1, y_2) \succsim (x_1, x_2)$$

Indeed, note that $(x_1, x_2) \not\succsim (y_1, y_2)$ can be expressed as

$$(x_1, x_2) \not\succsim (y_1, y_2) \text{ if } \begin{cases} y_1 \geq x_1, \text{ and if} \\ y_1 \neq x_1 \text{ or } y_2 > x_2 \end{cases} \tag{2}$$

Expression (2) describes that bundle (y_1, y_2) contains weakly more units of good 1 than (x_1, x_2) does, thus implying that a consumer with a lexicographic preference relation weakly prefers (y_1, y_2) to (x_1, x_2), i.e., $(y_1, y_2) \succsim (x_1, x_2)$. Therefore, we have shown that $(x_1, x_2) \not\succsim (y_1, y_2)$ implies $(y_1, y_2) \succsim (x_1, x_2)$. Hence, the preference relation is complete.

2. *Transitivity.* Let us take three bundles (x_1, x_2),(y_1, y_2) and $(z_1, z_2) \in \mathbb{R}^2$ with $(x_1, x_2) \succsim (y_1, y_2)$:

$$(x_1, x_2) \succsim (y_1, y_2) \text{ if and only if } \begin{cases} x_1 > y_1, \text{ or if} \\ x_1 = y_1 \text{ and } x_2 \geqslant y_2 \end{cases}$$

and $(y_1, y_2) \succsim (z_1, z_2)$, that is

$$(y_1, y_2) \succsim (z_1, z_2) \text{ if and only if } \begin{cases} y_1 > z_1, \text{ or if} \\ y_1 = z_1 \text{ and } y_2 \geqslant z_2 \end{cases}$$

Hence, we need to check for transitivity in the four possible cases in which $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2)$.

a) If $x_1 > y_1$, and $y_1 > z_1$, then by the transitivity of the "greater than or equal" operator ($\geq$), we obtain $x_1 > z_1$. As we know that $x_1 > z_1$ implies

$(x_1, x_2) \succsim (z_1, z_2)$, then transitivity holds in this case.

b) If ($x_1 = y_1$ and $x_2 \geqslant y_2$) and ($y_1 = z_1$ and $y_2 \geqslant z_2$), then ($x_1 = z_1$ and $x_2 \geqslant z_2$). In addition, we know that ($x_1 = z_1$ and $x_2 \geqslant z_2$) implies $(x_1, x_2) \succsim (z_1, z_2)$, which validates transitivity.

c) If $x_1 > y_1$, and ($y_1 = z_1$ and $y_2 \geqslant z_2$), then $x_1 > z_1$. As we know that $x_1 > z_1$ implies $(x_1, x_2) \succsim (z_1, z_2)$. Transitivity holds in this case as well.

d) If $y_1 > z_1$ and ($x_1 = y_1$ and $x_2 \geqslant y_2$), then $x_1 > z_1$, and we know that $x_1 > z_1$ implies $(x_1, x_2) \succsim (z_1, z_2)$, entailing that transitivity holds in this case as well. We have then checked all four cases under which $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2)$ may arise, and in all of them we obtained $(x_1, x_2) \succsim (z_1, z_2)$, confirming that this preference relation is transitive. Therefore, since the preference relation is complete and transitive, we can conclude that it is rational.

(b) Show that the lexicographic preference relation $\succsim$ *cannot* be represented by a utility function $u : X \times Y \to \mathbb{R}$.

- Let us work by contradiction. So, let us suppose that there is a utility function $u(\cdot)$ representing this lexicographic preference relation $\succsim$. Then, for any $x_1 \in X$, the pair $(x_1, 1)$ is strictly preferred to the pair $(x_1, 0)$, i.e., $(x_1, 1) \succ (x_1, 0)$. If there is a utility function $u(\cdot)$ representing this preference relation, then we must have that

$$(x_1, 1) \succ (x_1, 0) \Longleftrightarrow u(x_1, 1) > u(x_1, 0)$$

On the other hand, from the Archimedean property, we know that we can pick a rational number $r(x_1)$ such that it lies in between $u(x_1, 1)$ and $u(x_1, 0)$.

$$u(x_1, 1) > r(x_1) > u(x_1, 0)$$

Let us take any $x_1, x_2 \in X$, and let us suppose without loss of generality that $x_1 > x_2$. Similarly to our above result, we then have that

$$u(x_2, 1) > r(x_2) > u(x_2, 0)$$

And since $x_1 > x_2$, we have that

$$u(x_1, 1) > r(x_1) > u(x_1, 0) > u(x_2, 1) > r(x_2) > u(x_2, 0)$$

which implies

$$r(x_1) > r(x_2)$$

Then, $r(\cdot)$ provides a one-to-one function from the set of real numbers, $\mathbb{R}$ (which is uncountable) to the set of rational numbers, $\mathbb{Q}$, which is countable. But this is a mathematical impossibility.[1] Thus, we conclude that there can be no utility function representing the lexicographic preferences when they are defined over a continuous set $X \times Y$, where $X = [0, 1]$ and $Y = [0, 1]$.

(c) Assume now that this preference relation is defined on a *finite* consumption set $X = X_1 \times X_2$, where $X_1 = \{x_{11}, x_{12}, ..., x_{1n}\}$ and $X_2 = \{x_{21}, x_{22}, ..., x_{2m}\}$. [*Hint*: You can define a function $N_i(x_{ij})$ as the number of elements in sequence X_i prior to element x_{ij}; that is,

$$N_i(x_{ij}) = \#\{y \in X_i | y < x_{ij}\}.$$

Then define a utility function

$$u(y_1, y_2) = mN_1(y_1) + N_2(y_2), \text{ where } m > 0,$$

and for any pair $(y_1, y_2) \in X_1 \times X_2$.]

1. Let us first define a function $N_i(x_{ij})$ as the number of elements in sequence X_i prior to element x_{ij}:

$$N_i(x_i) = \#\{y \in X_i | y < x_{ij}\}, \text{ where } X_i = \{x_{i1}, x_{i2}, ..., x_{in}\}.$$

Then, we define a utility function $u(y_1, y_2) = mN_1(y_1) + N_2(y_2)$ for any pair $(y_1, y_2) \in X_1 \times X_2$. In order to show that this utility function indeed represents the lexicographic preference relation (when consumption sets are finite), we need to show the usual two lines of implication:

$$(y_1, y_2) \succsim (z_1, z_2) \Longrightarrow u(y_1, y_2) \geq u(z_1, z_2), \text{ and}$$

$$(y_1, y_2) \succsim (z_1, z_2) \Longleftarrow u(y_1, y_2) \geq u(z_1, z_2)$$

2. Let us first show that $(y_1, y_2) \succsim (z_1, z_2) \Longrightarrow u(y_1, y_2) \geq u(z_1, z_2)$. In order to

[1] For a review of real and rational numbers, see, for instance, Simon and Blume's *Mathematics for Economists*, pp. 848-849

show this result, we need that

$$\begin{cases} y_1 > z_1, \text{ or} \\ y_1 = z_1 \text{ and } y_2 \geqslant z_2 \end{cases} \quad \text{implies } mN_1(y_1) + N_2(y_2) \geq mN_1(z_1) + N_2(z_2)$$

Hence, we first need to check if this inequality is satisfied when $y_1 > z_1$, and when ($y_1 = z_1$ and $y_2 \geqslant z_2$).

a) Let us first check if $y_1 > z_1$ implies $mN_1(y_1)+N_2(y_2) \geq mN_1(z_1)+N_2(z_2)$. Alternatively, we can rewrite this inequality as

$$m\underbrace{[N_1(y_1) - N_1(z_1)]}_{a} + \underbrace{[N_2(y_2) - N_2(z_2)]}_{b} \geq 0 \tag{1}$$

Let us analyze if this expression can ever be negative (we will examine the infimum values) by separately evaluating the infimum of terms (a) and (b). Regarding term (a), we know that, if $y_1 > z_1$,

$$\inf [N_1(y_1) - N_1(z_1)] = k - (k-1) = 1,$$

$$\text{since } N_1(y_1) > N_1(z_1) \text{ given that } y_1 > z_1$$

and hence, $\inf [m [N_1(y_1) - N_1(z_1)]] = m$. Thus, $m [N_1(y_1) - N_1(z_1)] \geq m$, and term (a) in expression (1) is always weakly above m. Let us now focus on term (b) of expression (1):[2]

$$\inf [N_2(y_2) - N_2(z_2)] = \inf N_2(y_2) - \sup N_2(z_2) = 0 - (m-1) = 1 - m$$

Intuitively, the result $\inf N_2(y_2) = 0$ implies that there are no elements prior to y_2 (that is, y_2 is the first term of the sequence); in contrast, $\sup N_2(z_2) = m - 1$ means that z_2 is the last element in the sequence of length m, and hence all other $m-1$ elements in the sequence were located prior to z_2. Hence, $N_1(y_1) - N_1(z_1) \geq 1 - m$, and thus term (b) in expression (1) always lies above $1 - m$. Combining the results of the first and second term of the infimum of expression (1), we can conclude that

$$m [N_1(y_1) - N_1(z_1)] + [N_2(y_2) - N_2(z_2)] \geq m - (1 - m) = 1$$

[2] Note that we are not imposing any conditions on y_2 and z_2, since we only assumed that $y_1 > z_1$.

which is clearly above 0. Recall that we needed to show that

$$m\left[N_1(y_1) - N_1(z_1)\right] + \left[N_2(y_2) - N_2(z_2)\right] \geq 0.$$

Therefore, $y_1 > z_1$ indeed implies $u(y_1, y_2) \geq u(z_1, z_2)$.

b) Let us now check that ($y_1 = z_1$ and $y_2 \geqslant z_2$) also implies $mN_1(y_1) + N_2(y_2) \geq mN_1(z_1) + N_2(z_2)$. Alternatively, we can rewrite this inequality as

$$m\left[N_1(y_1) - N_1(z_1)\right] + \left[N_2(y_2) - N_2(z_2)\right] \geq 0$$

First, note that $y_1 = z_1$ implies that $N_1(y_1) = N_1(z_1)$. Second, note that $y_2 \geqslant z_2$ implies that $N_2(y_2) \geq N_2(z_2)$. Therefore, the above inequality becomes

$$0 + \underbrace{\left[N_2(y_2) - N_2(z_2)\right]}_{\geq 0} \geq 0$$

which confirms what we needed to show. Hence, ($y_1 = z_1$ and $y_2 \geqslant z_2$) indeed implies $u(y_1, y_2) \geq u(z_1, z_2)$.

3. Let us now show the opposite direction of implication, i.e., $(y_1, y_2) \succsim (z_1, z_2) \Longleftarrow u(y_1, y_2) \geq u(z_1, z_2)$. First, note that if $u(y_1, y_2) \geq u(z_1, z_2)$, then it must be that $mN_1(y_1) + N_2(y_2) \geq mN_1(z_1) + N_2(z_2)$. Rearranging, we obtain

$$m\left[N_1(y_1) - N_1(z_1)\right] + \left[N_2(y_2) - N_2(z_2)\right] \geq 0$$

Then, note that this inequality can be positive for two different reasons: (1) because $N_1(y_1) > N_1(z_1)$, which implies $y_1 > z_1$; or because (2) $N_1(y_1) = N_1(z_1)$ and $N_2(y_2) \geq N_2(z_2)$, which implies $y_1 = z_1$ and $y_2 \geqslant z_2$. And we know that, by definition, these two cases describe the lexicographic preference relation

$$(y_1, y_2) \succsim (z_1, z_2) \text{ if and only if } \begin{cases} y_1 > z_1, \text{ or if} \\ y_1 = z_1 \text{ and } y_2 \geqslant z_2 \end{cases}$$

Hence, $(y_1, y_2) \succsim (z_1, z_2) \Longleftarrow u(y_1, y_2) \geq u(z_1, z_2)$. Since we have shown this implication in both directions, then we have confirmed that this utility function indeed represents the lexicographic preference relation

$$(y_1, y_2) \succsim (z_1, z_2) \Longleftrightarrow u(y_1, y_2) \geq u(z_1, z_2)$$